# The Canonical Structure of Bigravity* 

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#### Abstract

This work is motivated by an intention to make the theory of bigravity more comprehensible. Bigravity is a modification of the General Relativity (GR), maybe even the most natural one because it is based on the equivalence principle. The Hamiltonian formalism in tetrad variables transparently demonstrates the structure of bigravity


## 1 Introduction

Lagrangian of the bigravity is a sum of two GR Lagrangians formed of two spacetime metrics $f_{\mu \nu}, g_{\mu \nu}$ and a potential of their interaction discovered by de Rham, Gabadadze, and Tolley [1, 2]. The potential of bigravity simplifies when the action is expressed through tetrads [3], not metrics. Both two sets of lapse-and-shift variables appear linearly in the Hamiltonian and can be treated as Lagrange multipliers at primary constraints. As the theory is explicitly invariant only under diagonal diffeomorphisms of the spacetime manifold and diagonal rotations of the spatial triads, the number of arbitrary Lagrange multipliers is $7(1+3+3)$ the same is the number of the first class constraints. Other Lagrange multipliers provide 10 second class constraints. The compatibility of the primary second class constraints with the dynamical equations provides 10 new equations, where 6 of them are the so-called tetrad symmetry conditions, and the other 4 are equivalent to the second class constraints of the metric approach. One of these 4 constraints accompanied by a

[^0]corresponding primary constraint serves to remove the ghost degree of freedom. Three other constraints together with their (fixed by the compatibility conditions) Lagrange multipliers serve to supplement the two Hamiltonianlike constraints. This reorganization of constraints reproduces the results of the celebrated Hassan-Rosen transform [4]. From the geometrical viewpoint, it is interesting to notice that three bilinear combinations of the pair of triads corresponding to the pair of spatial metrics appear in the dRGT potential. One of these combinations is symmetric and therefore can be treated as a new (hybrid) spatial metric. Its role in the coupling to matter fields requires a detailed investigation. It is interesting to mention the correspondence of this combination to the spatial components of the geometric mean of the two spacetime metrics introduced by Kocic [5].

Below for spacetime coordinate indices running from 0 to 3, we use small Greek letters; for internal indices running from 1 to 3 , we use small Latin letters from the beginning of the alphabet. For spatial indices small letters from the middle of the alphabet are used, for internal indices running from 0 to 3 the capital Latin letters are used. We deal with metrics that have common timelike and spacelike vectors. The variables related to the metric $g_{\mu \nu}$ are marked by an upper bar.

## 2 Lagrangian, potential, and variables

From the mathematical point of view, GR looks much simpler when expressed in the geometrical language, i.e. in variables having an evident geometrical meaning. This is true also for the bigravity. The Lagrangian of bigravity is equal to the sum of two copies of the GR Lagrangian minus an interaction term called the potential

$$
\begin{align*}
\mathcal{L}^{(f)} & =\frac{1}{16 \pi G^{(f)}} \sqrt{-f} f^{\mu \nu} R_{\mu \nu}^{(f)}+\mathcal{L}_{M}^{(f)}\left(\psi^{A}, f_{\mu \nu}\right),  \tag{1}\\
\mathcal{L}^{(g)} & =\frac{1}{16 \pi G^{(g)}} \sqrt{-g} g^{\mu \nu} R_{\mu \nu}^{(g)}+\mathcal{L}_{M}^{(g)}\left(\phi^{A}, g_{\mu \nu}\right),  \tag{2}\\
\mathcal{L} & =\mathcal{L}^{(f)}+\mathcal{L}^{(g)}-\frac{m^{2}}{2 \kappa} \sqrt{-g} U\left(f_{\mu \nu}, g_{\mu \nu}\right) . \tag{3}
\end{align*}
$$

The diffeomorphism invariance requires

$$
\begin{equation*}
U\left(f_{\mu \nu}, g_{\mu \nu}\right)=U(\text { invariants of } \mathbf{Y}), \quad \text { where } \quad \mathrm{Y}=g^{-1} f \equiv g^{\mu \alpha} f_{\alpha \nu} \tag{4}
\end{equation*}
$$

The first formulation of the GR in the Hamiltonian language was given by Dirac [6]. If we apply Arnowitt-Deser-Misner [7] (ADM) variables (lapses $N, \bar{N}$, shifts $N^{i}, \bar{N}^{i}$ and induced metrics $\eta_{i j}, \gamma_{i j}$ ) and introduce a basis for spacetime tensors ( $n^{\alpha}, e_{i}^{\alpha}$ ) introduced by Kuchar̆ and York [10] (formed by one of metrics, let it be $f_{\mu \nu}$ ) we obtain

$$
\mathbf{Y}=g^{-1} f=u^{-2}\left(\begin{array}{cc}
-\left[n^{\mu} n_{\nu}\right] & u^{i}\left[n^{\mu} e_{\nu i}\right]  \tag{5}\\
u^{j}\left[e_{j}^{\mu} n_{\nu}\right] & \left(-u^{i} u^{j}+u^{2} \gamma^{i j}\right)\left[e_{i}^{\mu} e_{\nu j}\right]
\end{array}\right)
$$

where

$$
\begin{equation*}
u=\frac{\bar{N}}{N}, \quad u^{i}=\frac{\bar{N}^{i}-N^{i}}{N} \tag{6}
\end{equation*}
$$

A standard longstanding problem of the nonlinear massive gravity (and also bigravity) was the Boulware-Deser [8] ghost arising due to nonlinearity of $\sqrt{-g} U$ in the auxiliary variable $u$. The potential proposed by de Rham, Gabadadze, and Tolley [1, 2] (dRGT) is as follows

$$
\begin{equation*}
U=\sum_{n=0}^{4} \beta_{n} e_{n}(X), \quad \mathrm{X}=\sqrt{\mathrm{Y}}, \quad \mathrm{Y}=\left\|g^{\mu \alpha} f_{\alpha \nu}\right\| \tag{7}
\end{equation*}
$$

where the symmetric polynomials of matrix $X_{\nu}^{\mu}={\sqrt{\left\|g^{-1} f\right\|^{\prime}}}_{\nu}^{\mu}$ written through traces of it and its powers are the following

$$
\begin{aligned}
e_{0} & =1 \\
e_{1} & =\operatorname{Tr} X \\
e_{2} & =\frac{1}{2}\left((\operatorname{Tr} X)^{2}-\operatorname{Tr} X^{2}\right) \\
e_{3} & =\frac{1}{6}\left((\operatorname{Tr} X)^{3}-3 \operatorname{Tr} X \operatorname{Tr} X^{2}+2 \operatorname{Tr} X^{3}\right), \\
e_{4} & =\operatorname{det} X
\end{aligned}
$$

Then a solution of the theory equations is given should give two spacetime metric tensors and all the matter fields.

## 3 Kuchař's notations

The Hamiltonian formalism of GR becomes more transparent when given in the embedding variables, i.e. as a dynamics of hypersurfaces. The suitable
variables are the induced metric $\gamma_{i j}$ and the external curvature tensor $K_{i j}$. In the ADM variables, the time components of the metric $g_{0 \mu}$ are replaced by the lapse and shift variables $N, N^{i}$ that connects the close hypersurfaces. In the Kuchar approach [9] (see also York [10]) $N, N^{i}$ are components of the 4 -vector connecting observer positions on the closest hypersurfaces

$$
\begin{equation*}
N^{\alpha} \equiv \frac{\partial X^{\alpha}}{\partial t}=N n^{\alpha}+N^{i} e_{i}^{\alpha} \tag{8}
\end{equation*}
$$

where two coordinate frames $X^{\alpha}$ and $\left(\tau, x^{i}\right)$ are used. The embedding functions $e^{\alpha}\left(\tau, x^{i}\right)$ provide one-to-one map $X^{\alpha}=e^{\alpha}\left(\tau, x^{i}\right)$. The three tangential to a hypersurface vectors are $e_{i}^{\alpha}=\frac{\partial e^{\alpha}}{\partial x^{i}}$. In bigravity we have two unit normal vectors they are denoted as $n^{\alpha}, \bar{n}^{\alpha}$, and they satisfy equations:

$$
\begin{array}{ll}
g_{\mu \nu} \bar{n}^{\mu} \bar{n}^{\nu}=-1, & g_{\mu \nu} \bar{n}^{\mu} e_{i}^{\nu}=0, \\
f_{\mu \nu} n^{\mu} n^{\nu}=-1, & f_{\mu \nu} n^{\mu} e_{i}^{\nu}=0 .
\end{array}
$$

The canonical variables are the two induced metrics $\eta_{i j}=f_{\mu \nu} e_{i}^{\mu} e_{j}^{\nu}, \gamma_{i j}=$ $g_{\mu \nu} e_{i}^{\mu} e_{j}^{\nu}$ and the two external curvature tensors $K_{i j}=-e_{i}^{\alpha} n_{\alpha ; \beta} e_{j}^{\beta}$, and $\bar{K}_{i j}=$ $-e_{i}^{\alpha} \bar{n}_{\alpha ; \beta} e_{j}^{\beta}$. Two spacetime metrics in their local bases $\left(\bar{n}^{\alpha}, e_{i}^{\alpha}\right)$, and $\left(n^{\alpha}, e_{i}^{\alpha}\right)$ are

$$
\begin{equation*}
g_{\mu \nu}=-\bar{n}_{\mu} \bar{n}_{\nu}+\gamma_{i j} \bar{e}_{\mu}^{i} \bar{e}_{\nu}^{j}, \quad f_{\mu \nu}=-n_{\mu} n_{\nu}+\eta_{i j} e_{\mu}^{i} e_{\nu}^{j} \tag{9}
\end{equation*}
$$

## 4 Tetrads

The metric tensor is not a unique choice of a dynamical variable. There is another possibility of the geometrical description provided by a field of orthonormal bases or tetrads [11]. In bigravity, we have two such bases $F_{\mu}^{A}, E_{\mu}^{A}$ given at each spacetime point. The potential now can be expressed explicitly, i.e. it is possible to find a matrix square root of the mixed tensor $\mathrm{Y}_{\beta}^{\alpha}=g^{\alpha \mu} f_{\mu \beta}$. But the physical content of the metric and tetrad formulations is the same only if symmetry conditions for the tetrads are fulfilled. The vierbeins (or tetrads) are the square root of metric

$$
\begin{gather*}
g=E^{T} E, \quad g_{\mu \nu}=E_{\mu A} E_{\nu}^{A}  \tag{10}\\
g^{-1}=E^{-1}\left(E^{-1}\right)^{T}, \quad g^{\mu \nu}=E_{A}^{\mu} E^{A \nu} \tag{11}
\end{gather*}
$$

Then we can extract the square root of the matrix Y

$$
\begin{equation*}
X=\sqrt{g^{-1} f}=\sqrt{E^{-1}\left(E^{-1}\right)^{T} F^{T} F}=E^{-1} F^{T} \tag{12}
\end{equation*}
$$

if symmetry conditions are fulfilled

$$
\begin{equation*}
\left(F E^{-1}\right)^{T}=F E^{-1} \tag{13}
\end{equation*}
$$

There is a diagonal Lorentz symmetry generated by

$$
L_{A B}^{+}=\left(\begin{array}{cc}
0 & L_{0 b}^{+} \\
L_{a 0}^{+} & L_{a b}^{+}
\end{array}\right)
$$

We can sacrifice $L_{a 0}^{+}$to achieve the null tetrad gauge for $E_{A \mu}$

$$
\begin{equation*}
E_{0 \mu}=\bar{n}_{\mu} . \tag{14}
\end{equation*}
$$

Then the dynamical variables occur triads $e_{i}^{a}$. But we can apply this gauge only to one tetrad as the potential is invariant under diagonal tetrad rotations

$$
\begin{equation*}
F_{\mu}^{\prime A}=\Lambda_{B}^{A} F_{\mu}^{B}, \quad E_{\mu}^{\prime A}=\Lambda_{B}^{A} E_{\mu}^{B} \tag{15}
\end{equation*}
$$

In the article by Hinterbichler and Rosen [3], it was suggested to parametrize the additional degrees of freedom by adding an arbitrary boost transformation to the triad basis. The parametrization of a boost

$$
\Lambda_{B}^{A}=\left(\begin{array}{cc}
\varepsilon & \varepsilon v_{b} \\
\varepsilon v^{a} & \mathcal{P}_{b}^{a}
\end{array}\right), \quad \mathcal{P}_{b}^{a}=\delta_{b}^{a}+\frac{\varepsilon^{2}}{\varepsilon+1} v^{a} v_{b},
$$

allows taking the second tetrad $F_{A \mu}$ in the form

$$
\begin{equation*}
F_{\mu}^{A}=\Lambda_{B}^{A} \mathcal{F}_{\mu}^{B} \tag{16}
\end{equation*}
$$

where $\mathcal{F}_{\mu}^{B}$ is a second tetrad given in the time gauge. In this work, we take parameters of this boost as canonical variables and introduce new momenta conjugate to them. This is different from the approach taken in H-R. Therefore we get 21 pairs of canonically conjugate variables:

$$
\begin{equation*}
\left(e_{a i}, \pi_{a}^{i}\right),\left(\tilde{f}_{a i}, \Pi_{a}^{i}\right),\left(\tilde{v}_{i}, \Pi_{0}^{i}\right), \tag{17}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{f}_{a i}=\mathcal{P}_{a b} f_{b i}, \quad \tilde{v}_{i}=\tilde{f}_{a i} v_{a} \tag{18}
\end{equation*}
$$

The other variables are Lagrange multipliers $N, N^{i}, u, u^{i}, \lambda_{a b}^{+}, \lambda_{a b}^{-}, \lambda^{a}$. The Hamiltonian is as follows

$$
\begin{equation*}
\mathrm{H}=\int d^{3} x\left[N\left(\mathcal{R}^{\prime \prime}+u \mathcal{S}^{\prime}+u^{i} \mathcal{S}_{i}\right)+N^{i} \mathcal{R}_{i}+\lambda_{a b}^{+} L_{a b}^{+}+\lambda_{a b}^{-} L_{a b}^{-}+\lambda^{a} L_{a 0}\right] \tag{19}
\end{equation*}
$$

It is necessary to compare the approach used here with the earlier work by Alexandrov [12] and the preceding article [13]. There the formalism was developed for the two general tetrads and two general connections. The spatial triads appear as a solution to the second class constraints. These second class constraints arise because of applying the first order Palatini formalism where connections and tetrads are initially treated as independent variables. A common feature of both approaches is that the tetrad symmetry conditions appear as a consequence of the compatibility of the primary constraints with the Hamiltonian dynamics. But this is not the case for the earlier work [13]. The tetrad approach was also considered in article [14].

## 5 Implicit functions used in metric approach

It is impossible to express the dRGT potential as an explicit function of the metric variables, therefore implicit functions are used. After extracting lapses and shifts of both metrics and making a special transform of variables [4] it is possible to express the potential as a function of $3 \times 3$-matrix $D^{i}{ }_{j}$. This matrix is to be symmetrical

$$
\begin{equation*}
D^{i j}=D^{j i} \tag{20}
\end{equation*}
$$

and satisfy the following equation

$$
\begin{equation*}
\gamma^{i j}=D_{k}^{i} v^{k} D^{j}{ }_{m} v^{m}+\varepsilon^{-2} D^{i k} D_{k}^{j} . \tag{21}
\end{equation*}
$$

The above equations for $D^{i}{ }_{j}$ follow from Eq.(15) when the Hassan-Rosen transform of variables

$$
\begin{equation*}
u^{i}=v^{i}+u D_{j}^{i} v^{j}, \quad \varepsilon^{-1}=\sqrt{1-\eta_{i j} v^{i} v^{j}} . \tag{22}
\end{equation*}
$$

is applied. We start from a definition of $D^{i}{ }_{j}$ by the following formula

$$
\mathbf{X}=\sqrt{\mathbf{Y}}=\varepsilon u^{-1}\left(\begin{array}{cc}
-\left[n^{\mu} n_{\nu}\right] & v^{i}\left[n^{\mu} e_{\nu i}\right]  \tag{23}\\
v^{j}\left[e_{j}^{\mu} n_{\nu}\right]
\end{array} \begin{array}{c}
\left(-v^{i} v^{j}+\varepsilon^{-2} u D^{i j}\right)
\end{array} e_{i}^{\mu} e_{\nu j}\right] .
$$

After squaring matrix X and comparing the result with the previously obtained expression for matrix Y we obtain equations (21). Therefore $D^{i}{ }_{j}$ depends on $\eta_{i j}, \gamma_{i j}$ and $v^{i}$, indices of $D^{i}{ }_{j}$ are moved up and down by $\eta_{i j}$ and its inverse $\eta^{i j}$. After heavy calculations, it occurred possible to find expressions for derivatives of $D^{i}{ }_{j}$ with respect to canonical coordinates $\eta_{i j}, \gamma_{i j}$.

This allowed to calculate Poisson brackets of the potential with the other terms of the Hamiltonian and to get the constraints algebra [15].

In another approach [16, 17] the potential as a whole is considered as an implicit function of lapses, shifts, and induced metrics. It is shown that if this function fulfills the homogeneous Monge-Ampere equation in lapses and shifts then the theory is free of the Boulware-Deser ghost. Also, it is supposed that the rank of the corresponding matrix is equal to three. The important properties of the implicit solutions of the Monge-Ampere equation are given in [18].

By putting to zero variables $n^{i}$, and so discarding the Hassan-Rosen transformation, but preserving $D^{i}{ }_{j}$ one may arrive at the precursor theory for the Minimal Theory of Bigravity [19].

## 6 Algebra of constraints and degrees of freedom

In this section, we present the results of the Poisson brackets calculations. With the tetrad variables, we obtain that the Hassan-Rosen transform can be written as follows

$$
\begin{equation*}
u^{i}=v^{i}+u \bar{v}^{i}, \tag{24}
\end{equation*}
$$

where

$$
\begin{equation*}
v^{i}=f^{i a} v_{a}, \quad \bar{v}^{i}=e^{i a} v_{a} \tag{25}
\end{equation*}
$$

Then we introduce

$$
\begin{gather*}
\mathcal{S}=\mathcal{S}^{\prime}+\bar{v}^{i} \mathcal{S}_{i}  \tag{26}\\
\mathcal{R}=\mathcal{R}^{\prime \prime}+v^{i} \mathcal{S}_{i}+u \mathcal{S} \tag{27}
\end{gather*}
$$

For the 1 st class constraints $\mathcal{R}, \mathcal{R}_{i}, L_{a b}^{+}$we get

$$
\begin{aligned}
\{\mathcal{R}(x), \mathcal{R}(y)\} & =\left(\eta^{i k} \mathcal{R}_{k}+u u^{i} \mathcal{S}\right)(x) \delta_{, i}(x, y)-(x \leftrightarrow y), \\
\left\{\mathcal{R}_{i}(x), \mathcal{R}(y)\right\} & =\mathcal{R}(x) \delta_{, i}(x, y)+u_{, i} \mathcal{S} \delta(x, y), \\
\left\{\mathcal{R}_{i}(x), \mathcal{R}_{j}(y)\right\} & =\mathcal{R}_{j}(x) \delta_{, i}(x, y)-\mathcal{R}_{i}(y) \delta_{, j}(y, x),
\end{aligned}
$$

and

$$
\begin{aligned}
\left\{L_{a b}^{+}(x), L_{c d}^{+}(y)\right\} & =\left(\delta_{a c} L_{d b}^{+}+\delta_{b c} L_{a d}^{+}-\delta_{a d} L_{c b}^{+}-\delta_{b d} L_{a c}^{+}\right) \delta(x, y) \\
\left\{\mathcal{R}_{i}(x), L_{a b}^{+}(y)\right\} & =L_{a b}^{+}(x) \delta_{, i}(x, y) \approx 0 \\
\left\{\mathcal{R}(x), L_{a b}^{+}(y)\right\} & =0
\end{aligned}
$$

For the 2nd class constraints $\mathcal{S}, \Omega$ results are the following

$$
\begin{aligned}
\{\mathcal{S}(x), \mathcal{S}(y)\} & =\bar{v}^{i} \mathcal{S}(x) \delta_{, i}(x, y)-\bar{v}^{i} \mathcal{S}(y) \delta_{, i}(y, x) \\
\{\mathcal{R}(x), \mathcal{S}(y)\} & =\left(u^{i}+u \bar{v}^{i}\right) \mathcal{S}(x) \delta_{, i}(x, y)+\left(u\left(\bar{v}^{i} \mathcal{S}\right)_{, i}-\Omega\right) \delta(x, y) \\
\{\mathcal{S}(x), \Omega(y)\} & \neq 0
\end{aligned}
$$

The Hassan-Rosen transform may also be written as follows

$$
\begin{equation*}
\bar{N}^{i}=N^{i}+N v^{i}+\bar{N} \bar{v}^{i}, \tag{28}
\end{equation*}
$$

The constraints $L_{a b}^{-}, G_{c d}, L_{a 0}$ are second class as we get

$$
\begin{aligned}
\left\{L_{a b}^{-}(x), G_{c d}(y)\right\} & =\left[\delta_{a c} z_{(b d)}-\delta_{a d} z_{(c b)}-\delta_{b c} z_{(a d)}+\delta_{b d} z_{(c a)}\right] \\
& \times \delta(x, y) \neq 0 \\
\left\{L_{a 0}(x), \mathcal{S}_{i}(y)\right\} & =e \tilde{f}_{b i}\left[\beta_{1} \delta_{b a} e_{0}(z)+\beta_{2}\left(\delta_{b a} e_{1}(z)-z_{b a}\right)\right. \\
& \left.+\beta_{3}\left(\delta_{b a} e_{2}(z)+z_{b c} z_{c a}-z z_{b a}\right)\right] \delta(x, y) \neq 0 .
\end{aligned}
$$

where

$$
\begin{equation*}
z_{a b}=e_{a i} \tilde{f}^{i b}=z_{b a}, \quad \tilde{f}^{i b}=\mathcal{P}_{a b}^{-1} f^{i a}, \quad \mathcal{P}_{a b}^{-1}=\delta_{a b}-\frac{\varepsilon}{\varepsilon+1} v_{a} v_{b} \tag{29}
\end{equation*}
$$

## 7 Conclusion

The results of this work are summarized in Table 1 where the number of gravitational degrees of freedom is calculated according to the formula

$$
\begin{equation*}
\mathrm{DOF}=\frac{1}{2}\left(n-2 n_{f . c .}-n_{\text {s.c. }}\right) \tag{30}
\end{equation*}
$$

The advantages of the proposed approach are the following. The potential (and so the Hamiltonian) is linear in the lapses and shifts $N, \bar{N}, N^{i}, \bar{N}^{i}$. All the nondynamical functions are Lagrange multipliers. The tetrad symmetry conditions follow from the Dirac procedure. The first three of them appear as secondary constraints, and the other three as a fixing of the Lagrangian multiplier $u^{i}$. Therefore the Hassan-Rosen transform is derived, and not postulated. Neither implicit functions, nor Dirac brackets are involved in the calculations. The geometrical meaning of the coefficients standing in the algebra of constraints is uncovered.

|  | BiGrav (general) | BiGrav (dRGT) | BiGrav (vierbein) |
| :---: | :---: | :---: | :---: |
| $(q, p)$ | $\left(\gamma_{i j}, \pi^{i j}\right),\left(\eta_{i j}, \Pi^{i j}\right)$ | $\left(\gamma_{i j}, \pi^{i j}\right),\left(\eta_{i j}, \Pi^{i j}\right)$ | $\left(e_{i a}, \pi^{i a}\right),\left(\tilde{f}_{i a}, \Pi^{i a}\right)$ |
| $n$ | 24 | 24 | $\left(p_{i}, \Pi_{0}^{i}\right)$ |
| $n$ | $\mathcal{R}, \mathcal{R}_{i}$ | $\mathcal{R}, \mathcal{R}_{i}$ | 42 |
| 1st class | 4 | $\mathcal{R}, \mathcal{R}_{i}, L_{a b}^{+}$ |  |
| $n_{\text {f.c. }}$ | 4 | $\mathcal{S}, \Omega$ | 7 |
| 2nd class | - |  | $\mathcal{S}, \Omega, L_{a b}^{-}, G_{a b}$ |
|  |  | 2 | $L_{a 0}, \mathcal{S}_{i}$ |
| $n_{\text {s.c. }}$ | 0 | 7 | 14 |
| DoF | 8 | 7 |  |

Table 1: The variables, constraints, and degrees of freedom.

We hope that the obtained results may be applied to the Cauchy problem, the perturbation theory, the numerical bigravity, the canonical quantization of bigravity.

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