

Geometry of Bigravity

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Abstract: The non-Euclidean geometry created by Bolyai, Lobachevsky and Gauss has led to a new physical theory—general relativity. In due turn, a correct mathematical treatment of the cosmological problem in general relativity has led Friedmann to a discovery of dynamical equations for the universe. And now, after almost a century of theoretical and experimental research, cosmology has a status of the most rapidly developing fundamental science. New challenges here are problems of dark energy and dark matter. As a result, a lot of modifications of general relativity appear recently. The bigravity is one of them, constructed with a couple of interacting space–time metrics accompanied by some coupling to matter. We discuss here this approach and different kinds of the coupling.

Keywords: non-euclidean geometry; general relativity; cosmology; bimetric gravity

1. Introduction

The non-Euclidean geometry was created by Bolyai, Lobachevski and Gauss in 19th century. The most impressive product of this discovery has appeared in 1915 as the general relativity theory created by Einstein with considerable impact of the two mathematicians, Grossmann and Hilbert [1–3]. The most important, unexpected and miraculous prediction of general relativity was obtained in 1922 in Petrograd by Friedmann [4]. It was discovered that the universe as a whole is a dynamical object subordinated to evolutionary equations [5]. It is mathematics that demonstrated its absolute and great creative power in the above chain of events.

In the 17th century, Blaise Pascal, mathematician, physicist and philosopher, had come to a sceptic view: “For after all what is man in nature? A nothing in regard to the infinite, a whole in regard to nothing, a mean between nothing and the whole; infinitely removed from understanding either extreme. The end of things and their beginnings are invincibly hidden from him in impenetrable secrecy...” [6] Now we can stay to be sceptics in relation to the search of the Theory of Everything, but we see that physics has made a great progress in both directions: to the infinity of small down to 10^{-23} km by means of LHC, and to 4×10^{-21} km by LIGO and Virgo, and to the infinity of large up to 10^{23} km by means of astrophysical instruments. The second number corresponds to the amplitude of gravitational waves registered by LIGO and Virgo, and the first to the length corresponding to LHC collision energy. The third number measures the horizon of the universe.

The first-rank problems of cosmology in the 21st century are dark energy and dark matter. Many theorists propose modifications of general relativity to deal with these enigmas and also, of course, with the cosmological singularity. One of these new approaches appearing in 2011 is the theory of bigravity. Here we will demonstrate some applications of this theory to cosmology.

2. Bigravity and Cosmology

The starting point was a problem of constructing a self-interacting massive spin-2 field theory. It was stated long ago by Deser and Boulware [7] that any version of such a theory should have a ghost degree of freedom. Nevertheless, recently, de Rham, Gabadadze and Tolley [8] (dRGT) have found

a potential, that is, a function of the two metric-like tensors $g_{\mu\nu}$ and $f_{\mu\nu}$ which provides a ghost-free theory. The dRGT potential is formed by means of symmetric polynomials of matrix $X^\mu_\nu = \left(\sqrt{g^{-1}f}\right)^\mu_\nu$ given through eigenvalues of X :

$$e_0 = 1, \tag{1}$$

$$e_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \tag{1}$$

$$e_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_4\lambda_1 + \lambda_1\lambda_3 + \lambda_2\lambda_4, \tag{2}$$

$$e_3 = \lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4, \tag{3}$$

$$e_4 = \lambda_1\lambda_2\lambda_3\lambda_4, \tag{4}$$

$$U = \frac{2m^2}{\kappa} \sqrt{-g} \sum_{n=0}^4 \beta_n e_n(X) = \frac{m^2}{8\pi G} \left(\beta_0 \sqrt{-g} + \dots + \beta_4 \sqrt{-f} \right). \tag{5}$$

Hassan and Rosen [9] have found a non-perturbative proof that the dRGT massive gravity is ghost-free, and generalized this proof for a new theory, called bigravity. We outline the construction of this theory below. Let both space–time metrics f and g be dynamical, and the Lagrangian is as follows,

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_g + \mathcal{L}_m - \sqrt{-g}U(f_{\mu\nu}, g_{\mu\nu}), \tag{6}$$

where \mathcal{L}_m is a matter Lagrangian, and \mathcal{L}_f and \mathcal{L}_g are two copies of the Hilbert–Einstein Lagrangian for metrics f and g . In the general case it is hard to deal with such a potential function, but fortunately this is not the case for homogeneous isotropic cosmology. Let us take the flat space cosmological ansatz for both metrics,

$$f_{\mu\nu} = (-N^2(t), R_f^2(t)\delta_{ij}), \quad g_{\mu\nu} = (-\bar{N}^2(t), R^2(t)\delta_{ij}), \tag{7}$$

then a couple of new variables appear,

$$u = \frac{\bar{N}}{N}, \quad r = \frac{R_f}{R}, \tag{8}$$

and the matrices Y and X occur diagonally,

$$Y^\mu_\nu = \left(g^{-1}f\right)^\mu_\nu = g^{\mu\alpha}f_{\alpha\nu} = \text{diag}\left(u^{-2}, r^2\delta_{ij}\right). \tag{9}$$

The positive square root of matrix Y is given below,

$$X = \sqrt{Y} = \text{diag}\left(+\sqrt{u^{-2}}, +\sqrt{r^2}\delta_{ij}\right) \equiv \text{diag}\left(u^{-1}, r\delta_{ij}\right), \tag{10}$$

and now λ_i and e_i are as follows

$$\lambda_1 = u^{-1}, \quad \lambda_2 = \lambda_3 = \lambda_4 = r, \tag{11}$$

$$e_0 = 1, \tag{12}$$

$$e_1 = u^{-1} + 3r, \tag{13}$$

$$e_2 = 3ru^{-1} + 3r^2, \tag{14}$$

$$e_3 = 3r^2u^{-1} + r^3, \tag{15}$$

$$e_4 = r^3u^{-1}. \tag{16}$$

The potential becomes linear in u ,

$$U = \frac{2m^2}{\kappa} N (uV + W), \tag{17}$$

where

$$V = \frac{1}{N} \frac{\partial U}{\partial u} = R^3 B_0(r), \tag{18}$$

$$W = \frac{1}{N} \left(U - u \frac{\partial U}{\partial u} \right) = R^3 B_1(r) \equiv R_f^3 \frac{B_1(r)}{r^3}, \tag{19}$$

where we apply deformed formulas for $(1+r)^3$

$$B_i(r) = \beta_i + 3\beta_{i+1}r + 3\beta_{i+2}r^2 + \beta_{i+3}r^3. \tag{20}$$

There are different possibilities for coupling matter to the spin-2 fields considered in detail, for example, in Ref. [10]:

1. one matter minimally couples to $g_{\mu\nu}$ (no BD ghost at all);
2. g-matter and f-matter minimally couple to $g_{\mu\nu}$ and $f_{\mu\nu}$ (no BD ghost at all);
3. one matter minimally couples to both $g_{\mu\nu}$ and $f_{\mu\nu}$ (BD ghost is present);
4. one matter minimally couples to “the effective metric” [11] (no BD ghost below the cut-off).

Here we restrict our treatment to the last case which is rather general. Of course, the first one is relatively simpler, but still very rich in solutions. The effective metric is constructed by means of the two metric tensors, or by means of the corresponding vierbeins

$$\mathcal{G}_{\mu\nu} = g_{\mu\nu} + 2\beta g_{\mu\alpha} \sqrt{g^{-1}f}^\alpha_\nu + \beta^2 f_{\mu\nu} = \left(E_\mu^A + \beta F_\mu^A \right) \left(E_{A\nu} + \beta F_{A\nu} \right), \tag{21}$$

and the interaction to matter is minimal; for example, for the scalar field we have

$$\mathcal{L}_\phi = \sqrt{-\mathcal{G}} \left(\frac{1}{2} \mathcal{G}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - U(\phi) \right). \tag{22}$$

Cosmological ansatz is as follows,

$$\mathcal{G}_{00} = -\mathcal{N}^2, \quad \mathcal{G}_{ij} = a^2 \delta_{ij}, \tag{23}$$

$$\mathcal{N} = N(u + \beta), \quad a = R + \beta R_f. \tag{24}$$

$$\sqrt{-\mathcal{G}} = \mathcal{N} a^3. \tag{25}$$

We obtain for the scalar field

$$\mathcal{L}_\phi = \mathcal{N} a^3 \left(\frac{1}{2} \left(\frac{\dot{\phi}}{\mathcal{N}} \right)^2 - U(\phi) \right), \quad \pi_\phi = \frac{a^3}{\mathcal{N}} \dot{\phi}. \tag{26}$$

Then, the primary constraints of the Hamiltonian formalism are as follows [10] (with a new notation $\mu = G_f/G_g$),

$$\mathcal{S} = \frac{3R^3}{8\pi G_g} \left[-H_g^2 + (1 + \beta r)^3 \frac{8\pi G_g \rho}{3} + \frac{m^2}{3} B_0(r) \right] = 0, \tag{27}$$

$$\mathcal{R}' = \frac{3R^3}{8\pi G_g} \left[-\frac{r^3 H_f^2}{\mu} + \beta (1 + \beta r)^3 \frac{8\pi G_g \rho}{3} + \frac{m^2}{3} B_1(r) \right] = 0. \tag{28}$$

Here, G_g and G_f denote gravitational constants corresponding to $g_{\mu\nu}$ and $f_{\mu\nu}$, and H_g and H_f are the corresponding Hubble parameters, ρ and p are energy density and pressure of an ideal fluid playing the role of matter. Next, there is a secondary constraint

$$\Omega = \frac{3R}{8\pi G_g} \Omega_1 \Omega_2 = 0, \tag{29}$$

$$\Omega_1 = rH_f - H_g, \tag{30}$$

$$\Omega_2 = \beta_1 + 2\beta_2 r + \beta_3 r^2 - \beta(1 + \beta r)^2 8\pi G_g p = 0, \tag{31}$$

which occurs factorized. For the first branch of solutions $\Omega_1 = 0$ the Friedmann equation for the observable Hubble constant,

$$H = \frac{\dot{a}}{\mathcal{N}a}, \tag{32}$$

is as follows:

$$H^2 = \frac{8\pi\tilde{G}\rho}{3} + \frac{\Lambda(r)}{3}, \quad \tilde{G} = (1 + \beta r)G. \tag{33}$$

The cosmological term and matter density are functions of r :

$$\Lambda(r) = m^2 \frac{B_0(r)}{(1 + \beta r)^2}, \tag{34}$$

$$\rho = \frac{m^2}{8\pi G} \frac{\frac{\mu B_1(r)}{r} - B_0(r)}{(1 + \beta r)^3 \left(1 - \frac{\mu\beta}{r}\right)}. \tag{35}$$

Then, the study of cosmological dynamics transforms into a study of dynamics for r (we suppose an equation of state $p = w\rho$),

$$\dot{r} = \frac{3NH a(1+w)(1+\beta r) \left(\frac{\mu B_1}{r} - B_0\right)}{B_0 - (B_{-1})' + \frac{\mu B_0'}{r} + \left(\frac{\mu B_1}{r} - B_0\right) \left(\frac{1}{1-\frac{\mu\beta}{r}} + \frac{3w}{1+\beta r}\right)}. \tag{36}$$

The critical points of this equations are as follows:

$$r = -\frac{1}{\beta}, \tag{37}$$

$$r = \mu\beta, \tag{38}$$

and the roots of quartic equation

$$\frac{\mu B_1(r)}{r} - B_0(r) = 0. \tag{39}$$

Let us mention that this model does not pretend to solve the problem of cosmological singularity. The infinities of the matter density or of the effective cosmological constant have the same meaning as in the standard Big Bang scenario, that is, they should be replaced by Planckian energy density. Moreover, $\rho = 0$ accompanied by nonzero Λ should be interpreted as de Sitter limit of the accelerated expansion. Due to the presence of a set of free parameters there are a lot of possibilities which should be investigated. Some solutions behave close to the Λ CDM model. Of course, one should provide the analysis of stability under local perturbations. It was argued [12] that a nice choice of parameters exists, because in the early universe the instabilities have limited times of existence.

3. Conclusions

Numerous publications have demonstrated that the dRGT bigravity has background solutions reproducing the standard cosmological model for the early universe. Simultaneously, this theory makes natural the presence of the cosmological constant in the contemporary universe evolution. By postulating the graviton mass value we can get the desirable value of $\Lambda \approx m^2$. Of course, this does

not solve the problem of cancelation of the large vacuum energy expected from the quantum field theory of matter. There are even some ideas on solving the dark matter problem on the base of bigravity [13]. The Hamiltonian approach to bigravity allows also to take the next move to study quantum cosmology models.

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Abbreviations

The following abbreviations are used in this manuscript:

LIGO Laser Interferometer Gravitational-Wave Observatory

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