Hamiltonian cosmology of bigravity

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The aim of this talk is to demonstrate that the Hamiltonian formalism of bigravity with de Rham, Gabadadze, Tolley (dRGT) potential reduced to minisuperspace is a flexible and powerful method to invent and to study cosmological models. We discuss here only the effective coupling of matter, i.e. the minimal interaction of it to a special combination of two dynamical metrics. It is shown that all the observables can be expressed as functions of the ratio of the two scale factors.

Keywords: Massive gravity; bimetric gravity; minisuperspace; canonical formalism.

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1. Introduction

The first treatment of cosmology as a dynamical problem was given by A. Friedmann in 1922[1]. Nobody had tried to apply the Hamiltonian formalism to his mechanical model of the Universe at that date. The canonical formulation as a rule demonstrates the internal structure of a theory (dynamical and auxiliary variables, constraint equations, degrees of freedom, statement of Cauchy problem) in the most clear way. Applying the ADM (Arnowitt-Deser-Misner) formalism to cosmology of General Relativity (GR) is now a popular method with a long history[2]. Recently much attention has been paid to cosmological models arising in the massive gravity and bigravity constructed from the dRGT (de Rham-Gabadadze-Tolley) potential. The exclusive place of this potential is due to the fact that it provides a violation of the BD (Boulware-Deser) ghost theorem[3] in finite range gravity. It was found that cosmological bigravity models naturally predict the appearance of dark energy in the Universe.

The aim of this talk is to demonstrate that the minisuperspace Hamiltonian formalism is a flexible and powerful method to invent and to study background cosmology models not only in GR, but in bigravity too. There are different ways to construct the coupling of matter to gravity in the presence of two or even more metric fields. We discuss here only one of them – the effective coupling of matter, i.e. the minimal interaction of it to a special combination of two dynamical metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ proposed in articles[4,5]. It was proved that this does not introduce the BD ghost below the relevant cut-off[4]. A complete treatment of background cosmology with different couplings in Hamiltonian language is given in other place[6]. The Lagrangian analysis of the effective coupling case is given in publicatons[7–9].

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2. Hamiltonian for flat space cosmology in GR

The Lagrangian of the gravitational field

$$\mathcal{L}_g = \frac{1}{16\pi G}\sqrt{-g}(g^{\mu\nu}R_{\mu\nu} - 2\Lambda) \tag{1}$$

accompanied by the matter, for example, scalar field Lagrangian

$$\mathcal{L}_\phi = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - U(\phi)\right), \tag{2}$$

provide us with GR equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - 2\Lambda) = 8\pi G T_{\mu\nu}. \tag{3}$$

In study of hthe omogeneous and isotropic cosmology we will use the Friedmann-Lemétre-Robertson-Walker (FLRW) ansatz. Let us take the spatially flat case for simplicity

$$g_{\mu\nu} = (-N^2(t), a^2(t)\delta_{ij}). \tag{4}$$

Then the Friedmann equation ("The Differential Equation by God") is as follows

$$\left(\frac{\dot a}{Na}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}. \tag{5}$$

The minisuperspace Lagrangian after omitting a total time derivative is the following

$$\mathcal{L} = Na^3\left(-\frac{3}{8\pi G}\left(\frac{\dot a}{Na}\right)^2 - \frac{\Lambda}{8\pi G} + p\right). \tag{6}$$

Then the Hamiltonian is

$$\mathrm{H} = Na^3\mathcal{R}, \tag{7}$$

where

$$H = \frac{\dot a}{Na}, \qquad \mathcal{R} \equiv -\frac{3H^2}{8\pi G} + \rho + \frac{\Lambda}{8\pi G} = 0. \tag{8}$$

N is a Lagrange multiple standing at the Hamiltonian constraint $\mathcal{R}$. Let us take the scale factor and the Hubble "constant" as Hamiltonian variables, then their Poisson bracket is as follows

$$\{a, H\} = -\frac{\kappa}{12a^2}. \tag{9}$$

The kinematical Hamiltonian equation is in fact a definition of $\mathcal{H}$

$$\dot a = \{a, \mathrm{H}\} = NHa, \tag{10}$$

whereas the dynamical equation is the following

$$\dot H = \{H, \mathrm{H}\} = -4\pi G N(\rho + p). \tag{11}$$

The conservation law comes from the matter Hamiltonian equations

$$\dot\rho + 3N\mathcal{H}(\rho + p) = 0. \tag{12}$$

3. Lagrangian formulation of bigravity

The Lagrangian density of bigravity may be written as follows

$$\mathcal{L} = \frac{1}{16\pi G}\sqrt{-g}R_g + \frac{1}{16\pi G_f}\sqrt{-f}R_f - \frac{m^2}{8\pi G}U(f_{\mu\nu}, g_{\mu\nu}) + \mathcal{L}_{matter}, \tag{13}$$

where two copies of GR gravitational Lagrangians are accompanied by a matter contribution and the dRGT potential. In cosmology we apply the FLRW ansatz for both metrics

$$f_{\mu\nu} = \Big(-N^2(t), \omega^2(t)\delta_{ij}\Big), \qquad \sqrt{-f} = N\omega^3, \tag{14}$$

$$g_{\mu\nu} = \Big(-N^2(t)u^2(t), \xi^2(t)\delta_{ij}\Big), \qquad \sqrt{-g} = Nu\xi^3, \tag{15}$$

and obtain rather simple expressions

$$\mathcal{L}_g = -\frac{3\xi^3 Nu}{8\pi G}\left(\frac{\dot{\xi}}{Nu\xi}\right)^2 \equiv -\frac{3\xi^3 Nu}{8\pi G}H_g^2, \qquad \mathcal{L}_f = -\frac{3\omega^3 N}{8\pi G_f}H_f^2, \tag{16}$$

where H_g and H_f are the two Hubble constants. In general case the dRGT potential is formed by means of the symmetric polynomials of matrix $\mathsf{X} = \sqrt{g^{-1}f}$:

$$U = \sqrt{-g}\sum_{n=0}^{4}\beta_n e_n(\mathsf{X}) = \beta_0\sqrt{-g} + \ldots + \beta_4\sqrt{-f} \equiv N\tilde{U}, \tag{17}$$

whereas in minisuperspace matrix $g^{-1}f$ is diagonal, and it is easy to find a square root of it

$$\mathsf{X} = \mathrm{diag}\left(+\sqrt{u^{-2}}, +\sqrt{r^2}\delta_{ij}\right) \equiv \mathrm{diag}\left(u^{-1}, r\delta_{ij}\right), \tag{18}$$

where $r = \omega/\xi$. Then the potential occurs a linear function of variable u:

$$U = \sqrt{-g}\sum_{i=0}^{4}\beta_i e_i = Nu\xi^3\sum_{i=0}^{4}\beta_i e_i = N\xi^3(uB_0(r) + B_1(r)), \tag{19}$$

where

$$B_i(r) = \beta_i + 3\beta_{i+1}r + 3\beta_{i+2}r^2 + \beta_{i+3}r^3. \tag{20}$$

The effective metric is constructed according to the formula[4]

$$\mathcal{G}_{\mu\nu} = g_{\mu\nu} + 2\beta g_{\mu\alpha}\mathsf{X}^\alpha_\nu + \beta^2 f_{\mu\nu}, \tag{21}$$

in tetrad language this is equivalent to exploiting a linear combination of the two tetrads:

$$\mathcal{G}_{\mu\nu} = h_{AB}\Big(E^A_\mu + \beta F^A_\mu\Big)\Big(E^B_\nu + \beta F^B_\nu\Big), \tag{22}$$

and in minisuperspace we have

$$\mathcal{G}_{\mu\nu} = \Big(-\mathcal{N}^2(t), a^2(t)\delta_{ij}\Big), \qquad \mathcal{N} = N(u+\beta), \qquad a = \xi(1+\beta r). \tag{23}$$

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As an example of matter, let us take the scalar field having a minimal interaction with the effective metric

$$\mathcal{L}_{matter} = \sqrt{-\mathcal{G}}\left(-\frac{1}{2}\mathcal{G}^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \mathcal{U}(\phi)\right). \tag{24}$$

4. Hamiltonian formalism of bigravity

We can also interprete the matter as an ideal fluid with density ρ and pressure p according to the following formulas

$$\mathcal{L}_{matter} = \mathcal{N}a^3 p, \qquad \mathcal{H}_{matter} = \mathcal{N}a^3\rho, \tag{25}$$

for the scalar field we have

$$\rho = \frac{\pi_\phi^2}{2a^6} + \mathcal{U}(\phi), \qquad p = \frac{\pi_\phi^2}{2a^6} - \mathcal{U}(\phi). \tag{26}$$

The minisuperspace Hamiltonian constructed from Eq.(13) is as follows

$$\mathrm{H} = N(\mathcal{R}' + u\mathcal{S}), \tag{27}$$

and the Poisson brackets are

$$\{\xi, H_g\} = -\frac{4\pi G}{3\xi^2}, \qquad \{\omega, H_f\} = -\frac{4\pi G_f}{3\omega^2}, \qquad \{\phi, \pi_\phi\} = 1. \tag{28}$$

The primary constraints are written below:

$$\mathcal{S} = \frac{3\xi^3}{8\pi G}\left[-H_g^2 + \frac{8\pi G\rho}{3}(1+\beta r)^3 + \frac{m^2}{3}B_0(r)\right], \tag{29}$$

$$\mathcal{R}' = \frac{3\omega^3}{8\pi G_f}\left[-H_f^2 + \frac{8\pi G_f\rho}{3}\beta\frac{(1+\beta r)^3}{r^3} + \frac{m^2}{3}\mu\frac{B_1(r)}{r^3}\right], \tag{30}$$

where $\mu = G_f/G$. The secondary constraint appears as a condition of consistency, and it has a factorized form

$$\Omega = \{\mathcal{S}, \mathcal{R}'\} = 3\left(\omega H_f - \xi H_g\right)\left(\frac{D_1(r)}{8\pi G} - \beta(1+\beta r)^2 p\right) \equiv \Omega_1\Omega_2, \tag{31}$$

where

$$D_i(r) = \beta_i + 2\beta_{i+1}r + \beta_{i+2}r^2. \tag{32}$$

There are two branches of solutions: $\Omega_1 = 0$ and $\Omega_2 = 0$. For the first branch $H_g = rH_f$, and we obtain the Friedmann equation for the observable Hubble constant $H = \dot{a}/(\mathcal{N}a)$

$$H^2 = \frac{8\pi G\rho}{3}(1+\beta r) + \frac{\Lambda(r)}{3}. \tag{33}$$

The cosmological term, the matter density and the observable Hubble constant itself may be treated as functions of the "hidden variable" r

$$\Lambda(r) = m^2\frac{B_0(r)}{(1+\beta r)^2}, \tag{34}$$

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$$\rho = \frac{m^2}{8\pi G} \frac{\frac{\mu B_1(r)}{r} - B_0(r)}{(1+\beta r)^3 \left(1 - \frac{\mu\beta}{r}\right)}, \tag{35}$$

$$H^2 = \frac{m^2}{3} \frac{B_1(r) - \beta B_0(r)}{(1+\beta r)^2 \left(\frac{r}{\mu} - \beta\right)}. \tag{36}$$

From kinematical relations we have

$$\dot{r} = \dot{a}(1 - ur)\frac{\alpha + \beta r}{\beta + \alpha u}, \tag{37}$$

whereas variable u is determined from the consistency of constraint Ω_1 evolution

$$\dot{\Omega}_1 = \{\Omega_1, \mathrm{H}\} \equiv N\left(\{\Omega_1, \mathcal{R}'\} + u\{\Omega_1, \mathcal{S}\}\right) = 0, \tag{38}$$

i.e.

$$u = -\frac{\{\Omega_1, \mathcal{R}'\}}{\{\Omega_1, \mathcal{S}\}}. \tag{39}$$

Then the study of cosmological dynamics transforms into a study of dynamics for r given by the following equation (we suppose equation of state $p = w\rho$ is valid)

$$\dot{r} = \frac{3NHa(1+w)(1+\beta r)\left(\frac{\mu B_1}{r} - B_0\right)}{B_0 - 3D_0 + \frac{3\mu D_1}{r} + \left(\frac{\mu B_1}{r} - B_0\right)\left(\frac{1}{1-\frac{\mu\beta}{r}} + \frac{3w}{1+\beta r}\right)}. \tag{40}$$

Here N is arbitrary in accordance with the freedom to make arbitrary time reparametrizations.

To solve the last equation and to interprete its solutions in terms of cosmological observables is a problem for numerical computer calculations. Zeros and poles of the functions $\rho(r)$, $\Lambda(r)$, $H(r)$ are of special interest. The critical points are

$$r = -\frac{1}{\beta}, \qquad r = \mu\beta,$$

and the roots of quartic equation

$$\frac{\mu B_1(r)}{r} - B_0(r) = 0.$$

There are free parameters: the dRGT potential coefficients β_i, the ratio of two gravitational constants μ and the mixing coefficient β, and the best set of them should be looked for. Stability of the background solutions is to be studied, and lifetimes of the arising instabilities are to be taken into account[10].

The second branch of solutions defined by equation

$$\frac{D_1(r)}{8\pi G} - \beta(1+\beta r)^2 p = 0, \tag{41}$$

has also been studied in recent work[11]. We should mention that the massive gravity case taking place when metric $f_{\mu\nu}$ is non-dynamical also leads to this equation[12].

5. Conclusion

The cosmology of bigravity is an open problem that is under active study[8–11]. This article is written in order to demonstrate the beauty and power of the Hamiltonian formalism in this field of research. We would like to mention that it is a direct road to develop the quantum cosmology of bigravity[13].

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