

Hamiltonian cosmology in bigravity and massive gravity

Vladimir O. Soloviev *

Institute for High Energy Physics,
National Research Center “Kurchatov Institute”

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Abstract

In the Hamiltonian language we provide a study of flat-space cosmology in bigravity and massive gravity constructed mostly with de Rham, Gabadadze, Tolley (dRGT) potential. It is demonstrated that the Hamiltonian methods are powerful not only in proving the absence of the Boulware-Deser ghost, but also in solving other problems. The purpose of this work is to give an introduction both to the Hamiltonian formalism and to the cosmology of bigravity. We sketch three roads to the Hamiltonian of bigravity with the dRGT potential: the metric, the tetrad and the minisuperspace approaches.

*E-mail: Vladimir.Soloviev@ihep.ru

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1 Introduction

One of the actual problems in modern physics is the problem of dark energy. There are different approaches to it, and one of them is to revise the standard theory of gravitation, i.e. the General Relativity (GR). New predictions are expected for physics at the very large distances, and they may be provided

by making gravity massive. The popular way to do so is to introduce a new tensor field with the same properties as the metric. From the aesthetic point of view it is natural to expect that this new field will be dynamical. Such a model is called bigravity or bimetric gravity.

The crucial problem in constructing the bigravity or massive gravity models is to avoid the Boulware-Deser ghost [1]. This problem was solved by de Rham, Gabadadze, and Tolley [2, 3] (dRGT), who proposed the genuine form of potential involving the matrix square root. The proof that this theory is ghost-free has been given first in the metric Hamiltonian approach [4, 5]. But it is hard to work with the dRGT potential in metric formalism, whereas in tetrad approach the calculations become explicit. Fortunately, the background cosmology that will be discussed in this work exploits only rather simple form for both the two metric tensors, and in the so-called minisuperspace approach the relevant matrix is diagonal, so the square root of it can be found directly.

Let us mention that the Boulware-Deser ghost appears below even in minisuperspace models for the two cases: first, when a non-dRGT potential is chosen (see Section 6), second, when one matter field minimally couples to a pair of metrics (see subsection 5.2).

A lot of work in cosmology of bigravity has been already done by many authors ¹. As a rule, the main instruments were the Lagrangian equations and Bianchi identities. The aim of this work is to demonstrate the main features and power of the Hamiltonian formalism in analyzing the variety of cosmological problems from a unique viewpoint. This variety arises due to many ways by which the matter fields can couple to the gravitational fields. We consider the minimal interactions of two species of matter with the two metrics, the double interactions of one matter with the two metrics, and the effective metric interacting with the matter. We also illustrate the general scheme by an example of the non-dRGT potential. For simplicity we consider only flat space dynamics, and each metric has only one matter source.

Here all the local physics is ignored including the important question of stability of the background cosmological solutions. This is discussed in many articles, for example, see recent publications [6, 7]. We are impressed by the statement [8] that it is possible to cure the growing perturbations by taking a tiny Planck mass for the second metric $f_{\mu\nu}$.

¹We are sorry for citing only part of them and not trying to give a complete list of references.

In Sections 2 and 3 we briefly explicate the construction of the bigravity Hamiltonian (first, in metric variables; second, in tetrad variables), for a technically more detailed presentation we address the reader to articles [9, 10, 11, 12, 13, 14]. For approaches to the Hamiltonian formalism that are not discussed here see [4, 5, 15, 16]. The title of this work should remind about a good old book [17]. The mini-superspace Hamiltonians are constructed in Section 4, the machinery given there is applied to the concrete derivations of the cosmological equations in Sections 5, 6.

2 Metric variables

The Lagrangian density of the bigravity theory may be written as follows

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_f + \mathcal{L}_M - \frac{2m^2}{\kappa} U(f_{\mu\nu}, g_{\mu\nu}),$$

where the two copies of the GR Lagrangian are involved

$$\mathcal{L}_g = \frac{1}{\kappa_g} \sqrt{-g} g^{\mu\nu} R_{\mu\nu}^{(g)}, \quad \mathcal{L}_f = \frac{1}{\kappa_f} \sqrt{-f} f^{\mu\nu} R_{\mu\nu}^{(f)}.$$

The matter Lagrangian and the potential for interaction of the two metrics are also included in \mathcal{L} . For simplicity, as an example of matter, we will take the scalar field with a minimal interaction with some metric $\mathcal{G}_{\mu\nu}$,

$$\mathcal{L}_\phi = \sqrt{-\mathcal{G}} \left(-\frac{1}{2} \mathcal{G}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \mathcal{U}(\phi) \right). \quad (2.1)$$

The role of metric below may be played by $g_{\mu\nu}$, or by $f_{\mu\nu}$, or by their combination called the effective metric (see subsection 5.3). The constant κ standing in front of the potential may be chosen coinciding with standard Newtonian one $\kappa_g = 16\pi G$, but in some works it is different, for example,

$$\frac{1}{\kappa} = \frac{1}{\kappa_g} + \frac{1}{\kappa_f}, \quad (2.2)$$

so we start with the general notation. We denote 3 + 1 components of a general metric $\mathcal{G}_{\mu\nu}$ and its inverse in the ADM coordinate basis as follows

$$\|\mathcal{G}_{\mu\nu}\| = \begin{pmatrix} -\mathcal{N}^2 + \psi_{mn} \mathcal{N}^m \mathcal{N}^n & \psi_{jk} \mathcal{N}^k \\ \psi_{ik} \mathcal{N}^k & \psi_{ij} \end{pmatrix}, \quad (2.3)$$

$$\|\mathcal{G}^{\mu\nu}\| = \begin{pmatrix} -\mathcal{N}^{-2} & \mathcal{N}^j \mathcal{N}^{-2} \\ \mathcal{N}^i \mathcal{N}^{-2} & \psi^{ij} - \mathcal{N}^i \mathcal{N}^j \mathcal{N}^{-2} \end{pmatrix}, \quad (2.4)$$

where \mathcal{N} is the lapse, \mathcal{N}^i is the shift vector, ψ_{ij} is the induced metric on spatial hypersurfaces of fixed time, and ψ^{ij} is the inverse matrix for it. The interaction potential of the two metrics suggested by de Rham, Gabadadze and Tolley [2, 3] is constructed as a linear combination of the symmetric polynomials e_i of matrix $\mathsf{X}_\nu^\mu = \left(\sqrt{g^{-1}f}\right)_\nu^\mu$

$$\begin{aligned}
e_0 &= 1, \\
e_1 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \\
e_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_4\lambda_1 + \lambda_1\lambda_3 + \lambda_2\lambda_4, \\
e_3 &= \lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4, \\
e_4 &= \lambda_1\lambda_2\lambda_3\lambda_4,
\end{aligned} \tag{2.5}$$

where λ_i are eigenvalues of X . Then

$$U = \sqrt{-g} \sum_{n=0}^4 \beta_n e_n(\mathsf{X}) = \beta_0 \sqrt{-g} + \dots + \beta_4 \sqrt{-f} \equiv N\tilde{U},$$

Our plan of constructing the Hamiltonian formalism is the following. First, we import the canonical variables and the expressions of Hamiltonians from the two copies of General Relativity

$$H_g = \int d^3x (\bar{N}\bar{\mathcal{H}} + \bar{N}^i\bar{\mathcal{H}}_i), \tag{2.6}$$

$$H_f = \int d^3x (N\mathcal{H} + N^i\mathcal{H}_i), \tag{2.7}$$

where

$$\bar{\mathcal{H}} = -\frac{1}{\sqrt{\gamma}} \left(\frac{1}{\kappa_g} \gamma R^{(3)} + \kappa_g \left(\frac{\pi^2}{2} - \text{Tr}\pi^2 \right) \right), \tag{2.8}$$

$$\bar{\mathcal{H}}_i = -2\pi_{i|j}^j, \tag{2.9}$$

$$\mathcal{H} = -\frac{1}{\sqrt{\eta}} \left(\frac{1}{\kappa_f} \eta R^{(\eta)} + \kappa_f \left(\frac{\Pi^2}{2} - \text{Tr}\Pi^2 \right) \right), \tag{2.10}$$

$$\mathcal{H}_i = -2\Pi_{i|j}^j. \tag{2.11}$$

The canonical gravitational variables have the standard Poisson brackets

$$\{\gamma_{ij}(x), \pi^{kl}(y)\} = \frac{1}{2} (\delta_i^k \delta_j^\ell + \delta_i^\ell \delta_j^k) \delta(x, y), \tag{2.12}$$

$$\{\eta_{ij}(x), \Pi^{kl}(y)\} = \frac{1}{2} (\delta_i^k \delta_j^\ell + \delta_i^\ell \delta_j^k) \delta(x, y). \tag{2.13}$$

The ADM-decompositions for the two space-time metrics are as follows

$$\|f_{\mu\nu}\| = \begin{pmatrix} -N^2 + \eta_{mn}N^mN^n & \eta_{jk}N^k \\ \eta_{ik}N^k & \eta_{ij} \end{pmatrix}, \quad (2.14)$$

$$\|f^{\mu\nu}\| = \begin{pmatrix} -N^{-2} & N^jN^{-2} \\ N^iN^{-2} & \eta^{ij} - N^iN^jN^{-2} \end{pmatrix}, \quad (2.15)$$

$$\|g_{\mu\nu}\| = \begin{pmatrix} -\bar{N}^2 + \gamma_{mn}\bar{N}^m\bar{N}^n & \gamma_{jk}N^k \\ \gamma_{ik}\bar{N}^k & \gamma_{ij} \end{pmatrix}, \quad (2.16)$$

$$\|g^{\mu\nu}\| = \begin{pmatrix} -\bar{N}^{-2} & \bar{N}^j\bar{N}^{-2} \\ \bar{N}^i\bar{N}^{-2} & \gamma^{ij} - \bar{N}^i\bar{N}^j\bar{N}^{-2} \end{pmatrix}, \quad (2.17)$$

Next we construct the matter Hamiltonian

$$H_M = \int d^3x (\mathcal{N}\mathcal{H}_M + \mathcal{N}^i\mathcal{H}_{Mi}),$$

and in case the role of matter is played by one scalar field with a minimal coupling to one metric $\mathcal{G}_{\mu\nu}$, according to Eqs.(2.3), (2.4) we obtain

$$\mathcal{H}_M = \frac{\pi_\phi^2}{2\sqrt{\psi}} + \frac{\sqrt{\psi}}{2}\psi^{ij}\partial_i\phi\partial_j\phi + \sqrt{\psi}\mathcal{U}(\phi).$$

In dealing with two or even more metrics instead of the coordinate basis it is suitable to use the geometrical basis [18, 19, 20, 21] formed by a unit normal to the state hypersurface n^α and three vectors tangential to this hypersurface e_i^α . Let X^μ be an arbitrary space-time coordinate system, and (t, x^i) is the ADM coordinate system, i.e. functions $X^\mu = X^\mu(t)$ give one-parametrical foliation of the space-time by spacelike hypersurfaces, whereas x^i are internal coordinates on a hypersurface. Then $e_i^\mu = \partial X^\mu / \partial x^i$ are the three tangential vectors. Of course, we need a space-time metric to construct the unit normal, let us suppose this metric to be $f_{\mu\nu}$, then

$$f_{\mu\nu}n^\mu n^\nu = -1, \quad f_{\mu\nu}n^\mu e_i^\nu = 0.$$

If we decompose over this basis another metric, for example, $g_{\mu\nu}$, we obtain

$$\|g^{\mu\nu}\| = \begin{pmatrix} -u^{-2}[n^\mu n^\nu] & u^j u^{-2}[n^\mu e_j^\nu] \\ u^i u^{-2}[e_i^\mu n^\nu] & (\gamma^{ij} - u^i u^j u^{-2})[e_i^\mu e_j^\nu] \end{pmatrix}, \quad (2.18)$$

$$\|g_{\mu\nu}\| = \begin{pmatrix} (-u^2 + \gamma_{mn}u^m u^n)[n_\mu n_\nu] & -\gamma_{jk}u^k[n_\mu e_\nu^j] \\ -\gamma_{ik}u^k[e_\mu^i n_\nu] & \gamma_{ij}[e_\mu^i e_\nu^j] \end{pmatrix}, \quad (2.19)$$

where new variables u , u^i appear. Their role is, at least, three-fold: first, they appear in the relations between two pairs of lapse and shift functions:

$$u = \frac{\bar{N}}{N}, \quad u^i = \frac{\bar{N}^i - N^i}{N}, \quad (2.20)$$

second, they are formed by projections of tensor $g^{\mu\nu}$ onto the basis (n_α, e_α^i) constructed with the metric $f_{\mu\nu}$

$$u = \frac{1}{\sqrt{-g^{\perp\perp}}} \equiv \frac{1}{\sqrt{-g^{\mu\nu}n_\mu n_\nu}}, \quad u^i = -\frac{g^{\perp i}}{g^{\perp\perp}} \equiv \frac{g^{\mu\nu}n_\mu e_\nu^i}{g^{\alpha\beta}n_\alpha n_\beta}, \quad (2.21)$$

third, they are the coefficients that connect the two bases $(\bar{n}_\alpha, \bar{e}_\alpha^i)$ and (n_α, e_α^i) :

$$\bar{n}_\mu = un_\mu, \quad \bar{e}_\mu^i = e_\mu^i - u^i n_\mu, \quad \bar{n}^\mu = \frac{1}{u}n^\mu - \frac{u^i}{u}e_i^\mu, \quad \bar{e}_i^\mu = e_i^\mu. \quad (2.22)$$

As the bases (n_α, e_α^i) and (n^α, e_i^α) are formed with the help of metric $f_{\mu\nu}$, this metric has only 6 nontrivial components in decompositions over them

$$\|f^{\mu\nu}\| = \begin{pmatrix} -[n^\mu n^\nu] & 0 \\ 0 & \eta^{ij}[e_i^\mu e_j^\nu] \end{pmatrix}, \quad (2.23)$$

$$\|f_{\mu\nu}\| = \begin{pmatrix} -[n_\mu n_\nu] & 0 \\ 0 & \eta_{ij}[e_\mu^i e_\nu^j] \end{pmatrix}. \quad (2.24)$$

One can see that all the components of matrix $Y_\nu^\mu = g^{\mu\alpha}f_{\alpha\nu}$ are functions of variables u , u^i , γ_{ij} and η_{ij} , the same statement is valid for the invariants of this matrix, for example, $\text{Tr}(Y^n)$. Therefore the eigenvalues of $\mathbf{X} = \sqrt{\mathbf{Y}}$ are also dependent only on these variables. In general case it is impossible to obtain the explicit expression for the dRGT potential in the form

$$U = N\tilde{U}(u, u^i, \eta_{ij}, \gamma_{ij}).$$

For this reason we start with a general function \tilde{U} and work with its formal derivatives

$$V = \frac{\partial \tilde{U}}{\partial u}, \quad (2.25)$$

$$V_i = \frac{\partial \tilde{U}}{\partial u^i}, \quad (2.26)$$

together with the function

$$W = \tilde{U} - u \frac{\partial \tilde{U}}{\partial u} - u^i \frac{\partial \tilde{U}}{\partial u^i}.$$

The Hamiltonian now appears in the following form

$$H = \int d^3x \left[N \left(\mathcal{H} + u \bar{\mathcal{H}} + u^i \bar{\mathcal{H}}_i + \mathcal{H}_M + \frac{2m^2}{\kappa} \tilde{U} \right) + N^i (\mathcal{H}_i + \bar{\mathcal{H}}_i + \mathcal{H}_{Mi}) \right].$$

Then the derivatives of the total Hamiltonian over non-dynamical variables N , N^i , u , u^i provide us with the primary relations:

$$\mathcal{R} = \frac{\partial H}{\partial N} \equiv \mathcal{H} + u \bar{\mathcal{H}} + u^i \bar{\mathcal{H}}_i + \mathcal{H}_M + \frac{2m^2}{\kappa} \tilde{U} = 0, \quad (2.27)$$

$$\mathcal{R}_i = \frac{\partial H}{\partial N^i} \equiv \mathcal{H}_i + \bar{\mathcal{H}}_i + \mathcal{H}_{Mi}, \quad (2.28)$$

$$\mathcal{S} = \frac{1}{N} \frac{\partial H}{\partial u} \equiv \bar{\mathcal{H}} + \frac{\partial \mathcal{H}_M}{\partial u} + \frac{2m^2}{\kappa} V = 0, \quad (2.29)$$

$$\mathcal{S}_i = \frac{1}{N} \frac{\partial H}{\partial u^i} \equiv \bar{\mathcal{H}}_i + \frac{\partial \mathcal{H}_M}{\partial u^i} + \frac{2m^2}{\kappa} V_i = 0. \quad (2.30)$$

Then we can rewrite the Hamiltonian as follows

$$H = \int d^3x [N (\mathcal{R}' + u \mathcal{S} + u^i \mathcal{S}_i) + N^i \mathcal{R}_i], \quad (2.31)$$

where

$$\mathcal{R}' = \mathcal{H} + \frac{2m^2}{\kappa} W, \quad (2.32)$$

For the potential \tilde{u} of the general form equations

$$\mathcal{S} = 0, \quad \mathcal{S}_i = 0, \quad (2.33)$$

may be solved for variables u , u^i , and then these solutions should be substituted into the expression of \mathcal{R}' in order to make it a constraint, i.e. a function of canonical variables only. After that equations

$$\mathcal{R} = 0, \quad \mathcal{R}_i = 0, \quad (2.34)$$

occur 1st class constraints arising as a consequence of the space-time coordinate invariance of the bigravity. On-shell, i.e. on the surface of variables satisfying Eqs.(2.33) they form the celebrated Poisson bracket algebra

$$\{\mathcal{R}(x), \mathcal{R}(y)\} \approx (\eta^{ik}(x) \mathcal{R}_k(x) + \eta^{ik}(y) \mathcal{R}_k(y)) \delta_{,i}(x, y), \quad (2.35)$$

$$\{\mathcal{R}_i(x), \mathcal{R}_k(y)\} \approx \mathcal{R}_i(y)\delta_{,k}(x, y) + \mathcal{R}_k(x)\delta_{,i}(x, y), \quad (2.36)$$

$$\{\mathcal{R}_i(x), \mathcal{R}(y)\} \approx \mathcal{R}(x)\delta_{,i}(x, y), \quad (2.37)$$

discussed by Dirac in the proposal to construct the field quantization on the curved surfaces [22]. For the potential \tilde{U} of the general form we obtain equations necessary to fulfil the above Poisson algebra

$$2\eta_{ik}\frac{\partial\tilde{U}}{\partial\eta_{jk}} + 2\gamma_{ik}\frac{\partial\tilde{U}}{\partial\gamma_{jk}} - u^j\frac{\partial\tilde{U}}{\partial u^i} - \delta_i^j\tilde{U} = 0, \quad (2.38)$$

$$2u^j\gamma_{jk}\frac{\partial\tilde{U}}{\partial\gamma_{ik}} - u^i u\frac{\partial\tilde{U}}{\partial u} + (\eta^{ik} - u^2\gamma^{ik} - u^i u^k)\frac{\partial\tilde{U}}{\partial u^k} = 0. \quad (2.39)$$

In order equation $\mathcal{S} = 0$ to become a genuine constraint excluding the Boulware-Deser ghost degree of freedom it is necessary to put one more condition on the potential. This condition is the requirement that it should be impossible to solve Eqs.(2.33) for auxiliary variables $u^a = (u, u^i)$, i.e. these equations are to be functionally dependent,

$$\frac{D(\mathcal{S}, \mathcal{S}_i)}{D(u, u^i)} = \det \left\| \frac{\partial^2}{\partial u^a \partial u^b} \left(\frac{2m^2}{\kappa} \tilde{U} + \mathcal{H}_M \right) \right\| \equiv \text{Hess}_{(4 \times 4)} \left(\frac{2m^2}{\kappa} \tilde{U} + \mathcal{H}_M \right) = 0. \quad (2.40)$$

Next we need to prove the existence of a secondary constraint Ω , and the fact that it has a non-zero Poisson bracket with \mathcal{S} . For this purpose it is useful to apply a method suggested by Fairlie and Leznov [23] on constructing implicit solutions of the homogeneous Monge-Ampère equation. This trick has been first developed and applied to massive gravity by Comelli, Nesti, and Pilo [11]. Combining Eqs.(2.38), (2.39) with the technique of Ref.[23] it is possible to prove that

$$\{\mathcal{S}(x), \mathcal{S}(y)\} \approx 0,$$

and therefore

$$\dot{\mathcal{S}} = \{\mathcal{S}, \mathbf{H}\} \approx \int d^3x N \{\mathcal{S}, \mathcal{R}'\} = \int d^3x N \Omega = 0 \quad \leftrightarrow \quad \Omega = 0.$$

By means of the Jacobi identity it is also possible to show that in general

$$\{\Omega, \mathcal{S}\} \neq 0,$$

and so these two constraints are second class in the Dirac terminology. The auxiliary variable u is fixed by the condition of preservation the secondary constraint in time

$$\{\Omega, \mathbb{H}\} = \int d^3x N (\{\Omega, \mathcal{R}'\} + u\{\Omega, \mathcal{S}\}) = 0.$$

The other auxiliary variables u^i are to be determined from equations $\mathcal{S}_i = 0$. For this purpose it is necessary to require

$$\det \left\| \frac{\partial \mathcal{S}_i}{\partial u^k} \right\| = \det \left\| \frac{\partial^2}{\partial u^i \partial u^k} \left(\frac{2m^2}{\kappa} \tilde{U} + \mathcal{H}_M \right) \right\| \equiv \text{Hess}_{(3 \times 3)} \left(\frac{2m^2}{\kappa} \tilde{U} + \mathcal{H}_M \right) \neq 0, \quad (2.41)$$

and therefore the rank of the big 4×4 Hessian (2.40) should be equal to three. Finally, we come to the Hamiltonian formalism of bigravity with 12 pairs of canonical variables (γ_{ij}, π^{ij}) , (η_{ij}, Π^{ij}) , four 1st class constraints $\mathcal{R}, \mathcal{R}_i$, and two 2nd class constraints \mathcal{S}, Ω . There are 7 gravitational degrees of freedom for such a gravitational potential.

Here we followed articles [9, 10]. For the massive gravity case see also [11].

3 Tetrad variables

It is possible also to construct the Hamiltonian approach to bigravity with the tetrad variables connected with the metrics by formulas

$$g^{\mu\nu} = E_A^\mu E_B^\nu h^{AB}, \quad f_{\mu\nu} = F_\mu^A F_\nu^B h_{AB}, \quad (3.1)$$

where h_{AB} is the Minkowski metric

$$h_{AB} = \text{diag}(-1, 1, 1, 1),$$

and (E_A^μ, E_μ^A) , (F_μ^A, F_A^μ) are the mutually inverse tetrad matrices. In general the tetrad Hamiltonian formalism is more involved than the metric one. As 10 components of the metric tensor are replaced by 16 tetrad components the number of constraint equations increases. Therefore the constraint algebra becomes more extended. In the same time the tetrad representation allows to obtain an explicit expression for the dRGT potential. Indeed, the matrix

$$\mathcal{X}_\nu^\mu = E^{\mu A} F_{\nu A}, \quad (3.2)$$

occurs a square root of the matrix $Y_\nu^\mu = g^{\mu\alpha} f_{\alpha\nu}$, if the symmetry equations

$$E_A^\mu F_\mu^B - E^{\mu B} F_{\mu A} = 0, \quad (3.3)$$

are valid. Then instead of Eqs.(2.5) it is suitable to apply equivalent formulas for the symmetric polynomials

$$\begin{aligned} e_0 &= 1, \\ e_1 &= \text{Tr}\mathbf{X}, \\ e_2 &= \frac{1}{2} ((\text{Tr}\mathbf{X})^2 - \text{Tr}\mathbf{X}^2), \\ e_3 &= \frac{1}{6} ((\text{Tr}\mathbf{X})^3 - 3\text{Tr}\mathbf{X}\text{Tr}\mathbf{X}^2 + 2\text{Tr}\mathbf{X}^3), \\ e_4 &= \frac{1}{24} ((\text{Tr}\mathbf{X})^4 - 6(\text{Tr}\mathbf{X})^2\text{Tr}\mathbf{X}^2 + 3(\text{Tr}\mathbf{X}^2)^2 + 8\text{Tr}\mathbf{X}\text{Tr}\mathbf{X}^3 - 6\text{Tr}\mathbf{X}^4) = \\ &= \det \mathbf{X} = \frac{\det ||F_{\mu a}||}{\det ||E_{\mu a}||} \equiv \frac{\sqrt{-f}}{\sqrt{-g}}. \end{aligned} \quad (3.4)$$

It is convenient to start with the privileged form of a tetrad, and a bit later transform it to the general form. Starting first with the metric $g_{\mu\nu}$ we propose to use as such a simplified tetrad the following set of four space-time vectors: the first vector E_0^α is the unit normal to the hypersurface of state \bar{n}^α constructed on the base of metric $g_{\mu\nu}$:

$$g_{\mu\nu} \bar{n}^\mu e_i^\nu = 0, \quad g_{\mu\nu} \bar{n}^\mu \bar{n}^\nu = -1,$$

the other three space-time vectors are the three tangential to the hypersurface vectors provided by the triads of the induced metric γ_{ij} . For this triads \mathbf{e}_i^a we have equations similar to Eqs.(3.1):

$$\gamma_{ij} = \mathbf{e}_i^a \mathbf{e}_j^b \delta_{ab}, \quad \delta_{ab} = \text{diag}(1, 1, 1), \quad \mathbf{e}_a^i \mathbf{e}_{ib} = \delta_{ab}, \quad \mathbf{e}_a^i \mathbf{e}_{ja} = \delta_j^i.$$

Then the tangential space-time vectors are formed as follows

$$E_a^\alpha = \mathbf{e}_a^i \bar{e}_i^\alpha, \quad \bar{e}_i^\alpha = e_i^\alpha \equiv \frac{\partial X^\alpha}{\partial x^i}.$$

The inverse tetrads are

$$E_\mu^0 = \bar{n}_\mu, \quad E_\mu^a = \bar{e}_\mu^i \mathbf{e}_{ia}, \quad \bar{e}_\mu^i \equiv g_{\mu\nu} e_j^\nu \gamma^{ji}.$$

Following this way for the metric $f_{\mu\nu}$ we also get a privileged tetrad

$$\mathcal{F}_\mu^0 = -n_\mu, \quad (3.5)$$

$$\mathcal{F}_\mu^a = e_\mu^j \mathbf{f}_{ja}, \quad e_\mu^j = f_{\mu\nu} e_i^\nu \eta^{ij}. \quad (3.6)$$

The general tetrad F_ν^A arises as a result of a boost transformation

$$\Lambda^A_B = \begin{pmatrix} \varepsilon & p_b \\ p^a & \mathcal{P}_b^a \end{pmatrix}, \quad \varepsilon = \sqrt{1 + p_a p^a}, \quad \mathcal{P}_b^a = \delta_b^a + \frac{1}{\varepsilon + 1} p^a p_b, \quad (3.7)$$

of the privileged one $F_\nu^A = \Lambda^A_B(p) \mathcal{F}_\nu^B$. Here

$$p^a = p_a, \quad h_{AB} \Lambda^A_C \Lambda^B_D = h_{CD}.$$

Then we can derive the matrix X_ν^μ defined in (3.2), and obtain the following

$$\mathsf{X}_\nu^\mu = \begin{pmatrix} A[n^\mu n_\nu] & B^j[n^\mu e_{\nu j}] \\ C^i[e_i^\mu n_\nu] & D^{ij}[e_i^\mu e_{\nu j}] \end{pmatrix}, \quad (3.8)$$

where

$$A = -\frac{\varepsilon}{u}, \quad (3.9)$$

$$B^j = \frac{p_a \mathbf{f}^{ja}}{u}, \quad (3.10)$$

$$C^i = \frac{u^i \varepsilon}{u} - p_a \mathbf{e}^{ia}, \quad (3.11)$$

$$D^{ij} = -\frac{u^i p^a \mathbf{f}^{ja}}{u} + \mathbf{f}^{ja} \mathcal{P}_{ab} \mathbf{e}^{ib}. \quad (3.12)$$

After straightforward calculation of all the symmetric polynomials (3.4) we get an explicit expression for the dRGT potential which occurs linear in variables u, u^i

$$\tilde{U} = uV + u^i V_i + W. \quad (3.13)$$

Functions $V, V_i,$ and W are given in Appendix. They depend on canonical variables $\mathbf{e}_{ia}, \mathbf{f}_{ia}$ and on auxiliary variable p_a . Especially, V_i occurs linear in p_a

$$V_i = -p_a C_{ab} \mathbf{f}_{ib}. \quad (3.14)$$

As the potential term does not contain any velocities, it enters the Hamiltonian of bigravity without any changes.

Next we are to find the corresponding Hamiltonian form for \mathcal{L}_g and \mathcal{L}_f . This is familiar from GR [24]. The new canonical variables are triads \mathbf{e}_{ia} , \mathbf{f}_{ia} together with their conjugate momenta π^{ia} , Π^{ia} , and their Poisson brackets are the following

$$\{\mathbf{f}_{ia}(x), \Pi^{jb}(y)\} = \{\mathbf{e}_{ia}(x), \pi^{jb}(y)\} = \delta_a^b \delta_i^j \delta(x, y), \quad (3.15)$$

$$\{\mathbf{f}_{ia}(x), \mathbf{e}_{jb}(y)\} = \{\pi^{ia}(x), \Pi^{jb}(y)\} = \{\mathbf{f}_{ia}(x), \mathbf{f}_{jb}(y)\} = 0, \quad (3.16)$$

$$\{\mathbf{e}_{ia}(x), \mathbf{e}_{jb}(y)\} = \{\pi^{ia}(x), \pi^{jb}(y)\} = \{\Pi^{ia}(x), \Pi^{jb}(y)\} = 0, \quad (3.17)$$

Momenta of the metric formalism are expressed through triad momenta as follows

$$\Pi^{ij} = \frac{1}{4} (\mathbf{f}^{ia} \Pi^{ja} + \mathbf{f}^{ja} \Pi^{ia}), \quad (3.18)$$

$$\pi^{ij} = \frac{1}{4} (\mathbf{e}^{ia} \pi^{ja} + \mathbf{e}^{ja} \pi^{ia}). \quad (3.19)$$

Then the Poisson brackets of the metric momenta are nonzero off-shell

$$\{\Pi^{ij}(x), \Pi^{kl}(y)\} = \frac{1}{4} (\eta^{ik} \mathcal{M}^{jl} + \eta^{il} \mathcal{M}^{jk} + \eta^{jk} \mathcal{M}^{il} + \eta^{jl} \mathcal{M}^{ik}), \quad (3.20)$$

$$\{\pi^{ij}(x), \pi^{kl}(y)\} = \frac{1}{4} (\gamma^{ik} \bar{\mathcal{M}}^{jl} + \gamma^{il} \bar{\mathcal{M}}^{jk} + \gamma^{jk} \bar{\mathcal{M}}^{il} + \gamma^{jl} \bar{\mathcal{M}}^{ik}). \quad (3.21)$$

Here the new constraints specific for the tetrad approach appear

$$\mathcal{M}^{ij} = \frac{1}{4} L_{ab} \mathbf{f}^{ja} \mathbf{f}^{ib} \equiv \frac{1}{4} (\mathbf{f}^{ia} \Pi_a^j - \mathbf{f}^{ja} \Pi_a^i) = 0, \quad (3.22)$$

$$\bar{\mathcal{M}}^{ij} = \frac{1}{4} \bar{L}_{ab} \mathbf{e}^{ja} \mathbf{e}^{ib} \equiv \frac{1}{4} (\mathbf{e}^{ia} \pi_a^j - \mathbf{e}^{ja} \pi_a^i) = 0, \quad (3.23)$$

or in a different form,

$$L_{ab} = \mathbf{f}_{ia} \Pi_b^i - \mathbf{f}_{ib} \Pi_a^i = 0, \quad (3.24)$$

$$\bar{L}_{ab} = \mathbf{e}_{ia} \pi_b^i - \mathbf{e}_{ib} \pi_a^i = 0. \quad (3.25)$$

The combined Hamiltonian, of course, contain these constraints

$$\begin{aligned} \mathbb{H}_{g+f} &= \mathbb{H}_g + \mathbb{H}_f = \\ &= \int d^3x (\bar{N} \bar{\mathcal{H}} + \bar{N}^i \bar{\mathcal{H}}_i + \bar{\lambda}^{ab} \bar{L}_{ab}) + \\ &+ \int d^3x (N \mathcal{H} + N^i \mathcal{H}_i + \lambda^{ab} L_{ab}), \end{aligned} \quad (3.26)$$

together with the constraints $\bar{\mathcal{H}}, \bar{\mathcal{H}}_i, \mathcal{H}, \mathcal{H}_i$ known from the metric Hamiltonian (2.8) – (2.11). As in bigravity the two metrics are mixed in the potential, only diagonal rotations of triads $\mathbf{e}_{ia}, \mathbf{f}_{ia}$ leave the Hamiltonian invariant. Therefore symmetric combinations

$$L_{ab}^+ = \bar{L}_{ab} + L_{ab} = 0, \quad (3.27)$$

become 1st class constraints, whereas constraints

$$L_{ab}^- = \bar{L}_{ab} - L_{ab} = 0, \quad (3.28)$$

are 2nd class.

Finally, we are to take into account the symmetry conditions (3.3). In the Hamiltonian variables they take the following form

$$G_a \equiv p_a + up_b \mathbf{f}_j^b \mathbf{e}_a^j - u^j \mathcal{P}_{ab} \mathbf{f}_j^b = 0, \quad (3.29)$$

$$G_{ab} \equiv \mathbf{f}_{ic} \mathcal{P}_{c[a} \mathbf{e}_{b]}^i = 0. \quad (3.30)$$

The Hamiltonian of bigravity now appears in the following form

$$\begin{aligned} \mathbb{H} &= \int d^3x N \left[\mathcal{H} + u \left(\bar{\mathcal{H}} + \frac{2m^2}{\kappa} V \right) + u^i \left(\bar{\mathcal{H}}_i + \frac{2m^2}{\kappa} V_i \right) + \frac{2m^2}{\kappa} W \right] + \\ &+ \int d^3x \left[N^i (\mathcal{H}_i + \bar{\mathcal{H}}_i) + \lambda_{ab}^+ L_{ab}^+ + \lambda_{ab}^- L_{ab}^- + \Lambda^a G_a + \Lambda^{ab} G_{ab} \right]. \end{aligned} \quad (3.31)$$

When two types of matter, f-matter and g-matter, separately and minimally couple to metrics $f_{\mu\nu}$ and $g_{\mu\nu}$, their contributions can be incorporated into constraints $\bar{\mathcal{H}}, \bar{\mathcal{H}}_i, \mathcal{H}, \mathcal{H}_i$. Besides the canonical variables the Hamiltonian depend on the Lagrangian multipliers $\lambda_{ab}^+, \lambda_{ab}^-, \Lambda^a, \Lambda^{ab}$, providing Eqs. (3.27)–(3.30) and on other auxiliary variables. Varying over u, u^i, N, N^i we obtain the following equations

$$\mathcal{S} = \bar{\mathcal{H}} + \frac{2m^2}{\kappa} V = 0, \quad (3.32)$$

$$\mathcal{S}_i = \bar{\mathcal{H}}_i + \frac{2m^2}{\kappa} V_i, \quad (3.33)$$

$$\mathcal{R} = \mathcal{R}' + u\mathcal{S} + u^i \mathcal{S}_i = 0, \quad \text{where} \quad \mathcal{R}' = \mathcal{H} + \frac{2m^2}{\kappa} W, \quad (3.34)$$

$$\mathcal{R}_i = \mathcal{H}_i + \bar{\mathcal{H}}_i = 0. \quad (3.35)$$

We do not vary over variable p_a that enter (3.31) in non-linear way. Eqs.(3.29) can be solved for u^i

$$u^i = \mathbf{f}^{ib} \left(\frac{p_b}{\varepsilon} + u p_a \mathbf{f}_j^a \mathbf{e}^{jc} \mathcal{P}_{cb}^{-1} \right), \quad (3.36)$$

where

$$\mathcal{P}_{cb}^{-1} = \delta_{cb} - \frac{p_c p_b}{\varepsilon(\varepsilon + 1)}. \quad (3.37)$$

Eqs. (3.33) contain variable p_a in a linear way and so can be easily solved for it

$$p_a = \frac{2\kappa}{m^2} \|C_{ab}\|^{-1} f^{ib} \bar{\mathcal{H}}_i. \quad (3.38)$$

Eqs.(3.30) depend on the canonical variables \mathbf{e}_b^i , \mathbf{f}_{ic} , and on the auxiliary variable p_a . After the exclusion of p_a G_{ab} become constraints. They have nonzero Poisson brackets with L_{ab}^- and together form 6 constraints of the 2nd class.

In the tetrad Hamiltonian formalism there are 18 pairs of the canonical variables $(\mathbf{e}_{ia}, \pi^{ia})$, $(\mathbf{f}_{ia}, \Pi^{ia})$, 7 constraints of the 1st class \mathcal{R}' , \mathcal{R}_i , L_{ab}^+ and 8 constraints of the 2nd class \mathcal{S} , Ω , L_{ab}^- , G_{ab} . Of course, we again have 7 gravitational degrees of freedom. The auxiliary variables p_a are found from equations $\mathcal{S}_i = 0$ which are linear in p_a . Variables u^i are found from equations $G_a = 0$ linear in u^i . The last auxiliary variable u can be found from

$$\dot{\Omega} = 0 = \int d^3x N (\{\Omega, \mathcal{R}'\} + u\{\Omega, \mathcal{S}\}), \quad (3.39)$$

this is also a linear equation.

The above exposition followed articles [12, 13]. Different treatments can be found in Refs. [14, 25, 26].

4 Minisuperspace

Let us look at the cosmological problem, i.e. consider the Hamiltonian evolution of a homogeneous and isotropic Universe in the bigravity theory. For simplicity and brevity only the spatially flat case will be treated. Most of attention will be paid to the theory with the dRGT potential.

In general, the bigravity Hamiltonian is a sum of kinetic and potential gravitational terms, composed of two metrics $g_{\mu\nu}$ and $f_{\mu\nu}$, plus one, or two matter contributions.

We take the Friedmann-Robertson-Walker ansatz for the two metrics

$$f_{\mu\nu} = (-N^2(t), \omega^2(t)\delta_{ij}), \quad \sqrt{-f} = N\omega^3, \quad (4.1)$$

$$g_{\mu\nu} = (-N^2(t)u^2(t), \xi^2(t)\delta_{ij}), \quad \sqrt{-g} = Nu\xi^3, \quad (4.2)$$

and introduce a new variable $r = \omega/\xi$.

4.1 Gravitational potential terms

For the admitted metric tensors of the form (4.1), (4.2) we obtain

$$g^{-1}f = g^{\mu\alpha}f_{\alpha\nu} = \text{diag}(u^{-2}, r^2\delta_{ij}). \quad (4.3)$$

Let us suppose

$$N > 0, \quad u > 0, \quad \xi > 0, \quad \omega > 0. \quad (4.4)$$

In fact, in calculating the matrix square root of the positive diagonal matrix (4.3) we consider only one (positive) square root for each positive expression ²

$$\mathbf{X} = \sqrt{g^{-1}f} = \text{diag}\left(+\sqrt{u^{-2}}, +\sqrt{r^2}\delta_{ij}\right) \equiv \text{diag}(u^{-1}, r\delta_{ij}), \quad (4.5)$$

and eigenvalues of this matrix are determined from the following equation

$$\det(\mathbf{X} - \lambda I) = 0. \quad (4.6)$$

It is easy to find that

$$\lambda_1 = u^{-1}, \quad \lambda_2 = \lambda_3 = \lambda_4 = r, \quad (4.7)$$

and then construct the symmetric polynomials:

$$e_0 = 1 \quad (4.8)$$

$$e_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \frac{1}{u} + 3r, \quad (4.9)$$

$$e_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = \frac{3r}{u} + 3r^2, \quad (4.10)$$

$$e_3 = \lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4 = \frac{3r^2}{u} + r^3, \quad (4.11)$$

$$e_4 = \lambda_1\lambda_2\lambda_3\lambda_4 = \frac{r^3}{u}. \quad (4.12)$$

²In this special case nothing serious will change if we take another sign for some eigenvalues.

The dRGT potential is a linear combination of these polynomials which occurs a linear function of variable u :

$$U = \sqrt{-g} \sum_{i=0}^4 \beta_i e_i = Nu \xi^3 \sum_{i=0}^4 \beta_i e_i = N(uV + W), \quad (4.13)$$

where

$$V = \beta_0 \xi^3 + 3\beta_1 \xi^2 \omega + 3\beta_2 \xi \omega^2 + \beta_3 \omega^3 \equiv \xi^3 B_0(r), \quad (4.14)$$

$$W = \beta_1 \xi^3 + 3\beta_2 \xi^2 \omega + 3\beta_3 \xi \omega^2 + \beta_4 \omega^3 \equiv \xi^3 B_1(r), \quad (4.15)$$

$$B_i(r) = \beta_i + 3\beta_{i+1}r + 3\beta_{i+2}r^2 + \beta_{i+3}r^3. \quad (4.16)$$

4.2 Gravitational kinetic terms

If we start with the Hilbert-Einstein gravitational Lagrangian density for the space-time metric $\mathcal{G}_{\mu\nu}$

$$\mathcal{L}_{\mathcal{G}} = \frac{1}{\kappa_{\mathcal{G}}} \sqrt{-\mathcal{G}} \mathcal{G}^{\mu\nu} R_{\mu\nu}, \quad (4.17)$$

and exploit the cosmological ansatz

$$\mathcal{G}_{\mu\nu} = (-\mathcal{N}^2(t), a^2(t)\delta_{ij}), \quad (4.18)$$

we get

$$\mathcal{L}_{\mathcal{G}} = \mathcal{T}_{\mathcal{G}} = -\frac{6a^3 \mathcal{N}}{\kappa} \left(\frac{\dot{a}}{\mathcal{N}a} \right)^2 \equiv -\frac{6a^3 \mathcal{N}}{\kappa} H^2, \quad (4.19)$$

where H is the Hubble constant.

Define the momentum canonically conjugate to a

$$\pi_a = \frac{\partial \mathcal{L}_{\mathcal{G}}}{\partial \dot{a}} = -\frac{12a}{\mathcal{N}\kappa} \dot{a} = -\frac{12a^2}{\kappa} H, \quad H = -\frac{\kappa \pi_a}{12a^2}, \quad \{a, \pi_a\} = 1. \quad (4.20)$$

It is suitable to use H instead of π_a , then we will use non-canonical Poisson bracket:

$$\{a, H\} = -\frac{\kappa}{12a^2}. \quad (4.21)$$

In bigravity we get the kinetic part of the Hamiltonian in the following form

$$H_{kinetic} = \mathcal{T}_f + \mathcal{T}_g = -\frac{6\omega^3 N}{\kappa_f} H_f^2 - \frac{6\xi^3 Nu}{\kappa_g} H_g^2. \quad (4.22)$$

This part generates the kinematical Hamiltonian equations

$$\dot{\omega} = \{\omega, \mathbf{H}_{kinetic}\} = N\omega H_f, \quad \dot{\xi} = \{\xi, \mathbf{H}_{kinetic}\} = Nu\xi H_g, \quad (4.23)$$

equivalent to the definition of two Hubble constants H_f , H_g . It is easy to derive an equation for the evolution of the relative scale factor r

$$\dot{r} = Nr(H_f - uH_g). \quad (4.24)$$

4.3 Matter terms

As an example of matter, we take the scalar field having a minimal interaction with gravity

$$\mathcal{L}_\phi = \sqrt{-\mathcal{G}} \left(-\frac{1}{2} \mathcal{G}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \mathcal{U}(\phi) \right), \quad (4.25)$$

where $\mathcal{G}_{\mu\nu}$ is the relevant metric of the form given by Eq.(4.18) (there will be different cases below), where $\phi_{,\mu} = \partial\phi/\partial x^\mu$. In the homogeneous cosmology $\phi = \phi(t)$ and

$$\mathcal{L}_\phi = \mathcal{T}_\phi[\dot{\phi}] - \mathcal{U}_\phi, \quad \mathcal{H}_\phi = \mathcal{T}_\phi[\pi_\phi] + \mathcal{U}_\phi. \quad (4.26)$$

Scalar fields may be of special interest in studying inflation in the bigravity [27]. If this matter field interacts with only one metric $\mathcal{G}_{\mu\nu}$, there is a straightforward correspondence between the velocity and the momentum

$$\mathcal{T}_\phi = \frac{a^3 \dot{\phi}^2}{2\mathcal{N}}, \quad \pi_\phi = \frac{\partial \mathcal{T}_\phi}{\partial \dot{\phi}} = \frac{a^3}{\mathcal{N}} \dot{\phi}, \quad \dot{\phi} = \frac{\mathcal{N}}{a^3} \pi_\phi. \quad (4.27)$$

We can also introduce density ρ and pressure p of matter according to the formulas

$$\mathcal{H}_\phi = \mathcal{N} a^3 \rho, \quad \rho = \frac{\pi_\phi^2}{2a^6} + \mathcal{U}(\phi), \quad p = \frac{\pi_\phi^2}{2a^6} - \mathcal{U}(\phi). \quad (4.28)$$

Below we will also consider the case when a scalar field simultaneously couples to metrics $g_{\mu\nu}$ and $f_{\mu\nu}$. Then

$$\mathcal{T}_\phi = \frac{\xi^3 \dot{\phi}^2}{2\bar{N}} + \frac{\omega^3 \dot{\phi}^2}{2N}, \quad \pi_\phi = \frac{\partial \mathcal{T}_\phi}{\partial \dot{\phi}} = \left(\frac{\xi^3}{u} + \omega^3 \right) \frac{\dot{\phi}}{N}, \quad \dot{\phi} = \frac{N\pi_\phi}{\frac{\xi^3}{u} + \omega^3}, \quad (4.29)$$

and therefore

$$\mathcal{T}_\phi = \frac{N\pi_\phi^2}{2\left(\frac{\xi^3}{u} + \omega^3\right)}, \quad \mathcal{U}_\phi = N(u\xi^3 + \omega^3)\mathcal{U}, \quad (4.30)$$

$$\mathcal{H}_\phi = N(u\xi^3 + \omega^3) \left(\frac{\pi_\phi^2}{2\left(\frac{\xi^3}{u} + \omega^3\right)(u\xi^3 + \omega^3)} + \mathcal{U}(\phi) \right). \quad (4.31)$$

Here \mathcal{H}_ϕ is a nonlinear function of u .

4.4 List of GR formulas given for reference

$$\mathcal{L}_g = \frac{1}{\kappa_g} \sqrt{-g} (g^{\mu\nu} R_{\mu\nu} - 2\Lambda) + \mathcal{L}_M(\phi^A, g_{\mu\nu}), \quad \kappa_g = 16\pi G, \quad (4.32)$$

$$\mathcal{L}_\phi = \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \mathcal{U}(\phi) \right), \quad (4.33)$$

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} (R - 2\Lambda) = \frac{\kappa_g}{2} T^{\mu\nu}, \quad T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}_M}{\partial g_{\mu\nu}}. \quad (4.34)$$

$$g_{\mu\nu} = (-\bar{N}^2(t), \xi^2(t) \delta_{ij}), \quad \sqrt{-g} = \bar{N} \xi^3, \quad \phi = \phi(t). \quad (4.35)$$

$$\mathcal{L}_g = \bar{N} \xi^3 \left(-\frac{6}{\kappa_g} \left(\frac{\dot{\xi}}{\bar{N} \xi} \right)^2 - 2\Lambda \right), \quad H_g = \frac{\dot{\xi}}{\bar{N} \xi}, \quad \pi_\xi = -\frac{12\xi^2}{\kappa_g} H_g. \quad (4.36)$$

$$\mathcal{L}_\phi = \bar{N} \xi^3 \left(\frac{1}{2} \left(\frac{\dot{\phi}}{\bar{N}} \right)^2 - \mathcal{U}(\phi) \right), \quad \pi_\phi = \frac{\xi^3}{\bar{N}} \dot{\phi}. \quad (4.37)$$

$$\rho = \frac{\pi_\phi^2}{2\xi^6} + \mathcal{U}(\phi) = \frac{1}{2} \left(\frac{\dot{\phi}}{\bar{N}} \right)^2 - \mathcal{U}(\phi), \quad p = \frac{\pi_\phi^2}{2\xi^6} - \mathcal{U}(\phi). \quad (4.38)$$

$$\mathbb{H} = \bar{N} \xi^3 \left(-\frac{6H_g^2}{\kappa_g} + \rho + \frac{2\Lambda}{\kappa_g} \right), \quad \{\xi, H_g\} = -\frac{\kappa_g}{12\xi^2}. \quad (4.39)$$

$$-\frac{6H_g^2}{\kappa_g} + \rho + \frac{2\Lambda}{\kappa_g} = 0, \quad \leftrightarrow \quad \left(\frac{\dot{\xi}}{\bar{N}\xi} \right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}. \quad (4.40)$$

$$\dot{\xi} = \bar{N}\xi H_g, \quad (4.41)$$

$$\dot{H}_g = -\frac{\bar{N}\kappa_g}{4}(\rho + p) = -4\pi G\bar{N}(\rho + p), \quad (4.42)$$

$$\dot{\rho} = -3\frac{\dot{\xi}}{\xi}(\rho + p). \quad (4.43)$$

If Λ and the matter equation of state $p = p(\rho)$ are known, the initial condition can be taken for ρ or for H_g . \bar{N} is arbitrary in accordance to the freedom of time reparametrization.

5 Matter couplings

5.1 f-matter and g-matter

The first works on cosmology of bigravity [28, 29, 30, 31] appeared rather soon after the proposal of the bigravity theory with the dRGT potential made by Hassan and Rosen [4]. In trying to solve the dark energy problem it is enough to admit g-matter only, but if also dark matter is under study then f-matter is also required [44].

If there are two species of matter: f-matter and g-matter, and each one has a minimal coupling to $f_{\mu\nu}$ or $g_{\mu\nu}$, then

$$\mathbb{H} = \mathbb{H}_{potential} + \mathbb{H}_{kinetic} + \mathbb{H}_{matter}, \quad (5.1)$$

where

$$\mathbb{H}_{potential} = \frac{2m^2}{\kappa} N(uV + W), \quad (5.2)$$

$$\mathbb{H}_{kinetic} = \frac{1}{\kappa_f} N(-6\omega^3 H_f^2) + \frac{1}{\kappa_g} Nu(-6\xi^3 H_g^2), \quad (5.3)$$

$$\mathbb{H}_{matter} = N\omega^3 \rho_f + Nu\xi^3 \rho_g. \quad (5.4)$$

The Hamiltonian can also be written as follows

$$\mathbb{H} = N\mathcal{R} \equiv N\mathcal{R}' + Nu\mathcal{S}, \quad (5.5)$$

where primary constraints are $\mathcal{R} \equiv \mathcal{R}' + u\mathcal{S}$ and \mathcal{S} , or better, \mathcal{R}' and \mathcal{S} :

$$\mathcal{R}' = -\frac{6}{\kappa_f}\omega^3 H_f^2 + \omega^3 \rho_f + \frac{2m^2}{\kappa}\xi^3 B_1(r) = 0, \quad (5.6)$$

$$\mathcal{S} = -\frac{6}{\kappa_g}\xi^3 H_g^2 + \xi^3 \rho_g + \frac{2m^2}{\kappa}\xi^3 B_0(r) = 0. \quad (5.7)$$

For the scalar sources we have

$$\rho_f = \frac{\Pi_\Phi^2}{2\omega^6} + \mathcal{U}_f(\Phi), \quad (5.8)$$

$$\rho_g = \frac{\pi_\phi^2}{2\xi^6} + \mathcal{U}_g(\phi). \quad (5.9)$$

The constraints (5.6), (5.7) may be written as a couple of Friedmann equations:

$$H_f^2 = \frac{\kappa_f}{6}\rho_f + \frac{\Lambda_f}{3}, \quad \Lambda_f = m^2 \frac{\kappa_f}{\kappa} \frac{B_1(r)}{r^3}, \quad (5.10)$$

$$H_g^2 = \frac{\kappa_g}{6}\rho_g + \frac{\Lambda_g}{3}, \quad \Lambda_g = m^2 \frac{\kappa_g}{\kappa} B_0(r). \quad (5.11)$$

We see that, in contrast to GR (4.40), the cosmological terms naturally appear in the bigravity, and they are not constants, but dynamical quantities. Hamiltonian equations for the matter fields provide us with a couple of conservation laws

$$\dot{\rho}_f + 3\frac{\dot{\omega}}{\omega}(\rho_f + p_f) = 0, \quad (5.12)$$

$$\dot{\rho}_g + 3\frac{\dot{\xi}}{\xi}(\rho_g + p_g) = 0. \quad (5.13)$$

The secondary constraint arises in order to preserve the primary constraint during the evolution

$$\dot{\mathcal{S}} = \{\mathcal{S}, \mathbf{H}\} = N\{\mathcal{S}, \mathcal{R}'\} \equiv N\Omega = 0, \quad (5.14)$$

where

$$\Omega \equiv \{\mathcal{S}, \mathcal{R}'\} = \frac{6m^2}{\kappa} (\omega H_f - \xi H_g) (\beta_1 \xi^2 + 2\beta_2 \xi \omega + \beta_3 \omega^2) = 0. \quad (5.15)$$

As the secondary constraint is factorized

$$\Omega = \Omega_1 \Omega_2, \quad (5.16)$$

there are two branches of solutions for it:

$$\Omega_1 = 0, \quad \leftrightarrow \quad H_g = r H_f, \quad (5.17)$$

and

$$\Omega_2 = 0, \quad \leftrightarrow \quad \beta_1 + 2\beta_2 r + \beta_3 r^2 = 0. \quad (5.18)$$

Below we consider both two cases.

Variable u is fixed by the following requirement

$$\dot{\Omega} = \{\Omega, H\} = N (\{\Omega, \mathcal{R}'\} + u\{\Omega, \mathcal{S}\}) = 0, \quad (5.19)$$

here we also have two cases:

$$u = -\frac{\{\Omega_1, \mathcal{R}'\}}{\{\Omega_1, \mathcal{S}\}}, \quad \text{or} \quad u = -\frac{\{\Omega_2, \mathcal{R}'\}}{\{\Omega_2, \mathcal{S}\}}. \quad (5.20)$$

For the two Hubble constants we get dynamical Hamiltonian equations

$$\dot{H}_f = \{H_f, H\} = \{H_f, \omega\} \frac{\partial H}{\partial \omega}, \quad (5.21)$$

$$\dot{H}_g = \{H_g, H\} = \{H_g, \xi\} \frac{\partial H}{\partial \xi}, \quad (5.22)$$

which appear in the following explicit form (with account for Eqs.(5.6), (5.7))

$$\dot{H}_f = -\frac{N\kappa_f}{4} \left[\rho_f + p_f + (1 - ur) \frac{2m^2 D_1(r)}{\kappa r^3} \right], \quad (5.23)$$

$$\dot{H}_g = -\frac{Nu\kappa_g}{4} \left[\rho_g + p_g - (1 - ur) \frac{2m^2 D_1(r)}{\kappa u} \right], \quad (5.24)$$

where

$$D_i(r) = \beta_i + 2\beta_{i+1}r + \beta_{i+2}r^2. \quad (5.25)$$

5.1.1 Massive gravity on Minkowskian background

If we suppose $f_{\mu\nu}$ to be a non-dynamical Minkowskian metric

$$N = 1, \quad \omega = 1, \quad r = \frac{1}{\xi}, \quad (5.26)$$

and, of course, exclude f-matter

$$\rho_f = 0, \quad p_f = 0, \quad (5.27)$$

then we get a massive gravity theory

$$H = u\mathcal{S} + \frac{2m^2}{\kappa}W, \quad (5.28)$$

$$\mathcal{S} = \xi^3 \left[-\frac{6}{\kappa_g}H_g^2 + \rho_g + \frac{2m^2}{\kappa}B_0 \left(\frac{1}{\xi} \right) \right] = 0, \quad (5.29)$$

$$\Omega = -\frac{12m^2}{\kappa}\xi H_g (\beta_1\xi^2 + 2\beta_2\xi + \beta_3) = 0. \quad (5.30)$$

From the last equation it follows that there are no non-static homogeneous and isotropic spatially flat cosmological solutions. This result has been obtained in Refs. [32, 33].

5.1.2 First branch of bigravity

From Eq.(5.17) we get a relation between the two Hubble constants and taking into account the Friedmann equations (5.10), (5.11) we can derive the equation relating the two matter energy densities

$$H_f = r^{-1}H_g, \quad (5.31)$$

$$\rho_g = \mu r^2 \rho_f + \frac{2m^2}{\kappa} \left[\mu \frac{B_1(r)}{r} - B_0(r) \right], \quad (5.32)$$

where

$$\mu = \frac{\kappa_f}{\kappa_g}. \quad (5.33)$$

The first of Eqs.(5.20) gives

$$u = \frac{\mu r^3(\rho_f + 3p_f) + \frac{2m^2}{\kappa}(\mu(B_1 - 3rD_2) + 3r^2D_1)}{r[r(\rho_g + 3p_g) + \frac{2m^2}{\kappa}(r(B_0 - 3D_0) + 3\mu D_1)]}. \quad (5.34)$$

Eq. (4.24) acquires the following form

$$\dot{r} = (1 - ru)NH_g, \quad (5.35)$$

and a fixed point for r appears at

$$r = \frac{1}{u}. \quad (5.36)$$

It corresponds to proportional space-time metrics $f_{\mu\nu} = r^2 g_{\mu\nu}$.

In general case for the expression $1 - ur$ we have the following formula

$$1 - ur = \frac{(\rho_g + 3p_g) - \mu r^2(\rho_f + 3p_f) + \frac{4m^2}{\kappa} \left(\frac{\mu B_1}{r} - B_0 \right)}{(\rho_g + 3p_g) + \frac{2m^2}{\kappa} \left(B_0 - 3D_0 + \frac{3\mu D_1}{r} \right)}. \quad (5.37)$$

In order to solve the system of Hamiltonian equations one can take as the initial data $\rho_g(t_0)$ and $H_g(t_0)$. Then the constraint \mathcal{S}

$$H_g^2 = \frac{\kappa_g}{6}\rho_g + \frac{\Lambda_g}{3}, \quad (5.38)$$

allows to find $\Lambda_g(t_0)$, and next, $r(t_0) = r_0$ can be found as one of the roots for the cubic equation

$$B_0(r_0) = \frac{\Lambda_g(t_0)}{m^2} \frac{\kappa}{\kappa_g}. \quad (5.39)$$

Then constraint Ω_1 :

$$H_f = r^{-1}H_g, \quad (5.40)$$

gives us H_f . The constraint \mathcal{R}' :

$$H_f^2 = \frac{\kappa_f}{6}\rho_f + \frac{\Lambda_f}{3}, \quad \Lambda_f = \mu m^2 \frac{\kappa_g}{\kappa} \frac{B_1(r)}{r^3}, \quad (5.41)$$

provides us with ρ_f . With account for Eq.(5.37) and equations of state $p_g = p_g(\rho)$, $p_f = p_f(\rho)$ it is possible to integrate the dynamical equations

$$\begin{aligned} \dot{r} &= (1 - ru)NH_g, \\ \dot{\rho}_g &= -3NuH_g(\rho_g + p_g), \\ \dot{\rho}_f &= -3NH_f(\rho_f + p_f), \\ \dot{H}_g &= -\frac{N\kappa_g}{4} \left[u(\rho_g + p_g) - (1 - ur) \frac{2m^2}{\kappa} D_1(r) \right], \\ \dot{H}_f &= -\frac{N\kappa_f}{4} \left[\rho_f + p_f + (1 - ur) \frac{2m^2}{\kappa} \frac{D_1(r)}{r^3} \right], \end{aligned}$$

where $N(t)$ is an arbitrary monotonic function responsible for a freedom of the time reparametrization.

Besides the general case, when $1 - ur$ is given by Eq.(5.37), it is tempting to consider the case when both types of matter have the same equation of state

$$p_g = w\rho_g, \quad p_f = w\rho_f, \quad (5.42)$$

then Eq.(5.37) is simplified, and is looking as follows

$$1 - ur = \frac{3(\frac{1}{2} + w) (\mu \frac{B_1}{r} - B_0)}{B_0 - 3D_0 + \frac{3\mu D_1}{r} + \frac{\kappa}{2m^2}(1 + 3w)\rho_g}. \quad (5.43)$$

5.1.3 Second branch of bigravity

In the second case (5.18) the value of variable r is a constant:

$$D_1(r) \equiv \beta_1 + 2\beta_2 r + \beta_3 r^2 = 0, \quad \leftrightarrow \quad r = \frac{-\beta_2 \pm \sqrt{\beta_2^2 - \beta_1\beta_3}}{\beta_3}. \quad (5.44)$$

Then from the kinematical equation (4.24) it follows

$$u = \frac{H_f}{H_g}, \quad (5.45)$$

The last equation is not a constraint, in contrast to (5.31), therefore the Friedmann equations (5.10), (5.11), and correspondingly, the dynamics for metrics $g_{\mu\nu}$, $f_{\mu\nu}$ decouple in this case. Dynamics for the two metrics are the same as in GR with the cosmological constants

$$\Lambda_g = m^2 \frac{\kappa_g}{\kappa} B_0(r), \quad \Lambda_f = \mu m^2 \frac{\kappa_g}{\kappa} \frac{B_1(r)}{r^3}. \quad (5.46)$$

Eqs.(5.23), (5.24) are as follows

$$\dot{H}_f = -\frac{N\kappa_f}{4} (\rho_f + p_f), \quad (5.47)$$

$$\dot{H}_g = -\frac{Nu\kappa_g}{4} (\rho_g + p_g), \quad (5.48)$$

and are the same as in GR (4.42). If we put ourselves into g-world, then the only artifact of the bigravity will be the fixed value of Λ_g .

The full set of equations consists of the algebraic ones

$$r = \frac{-\beta_2 \pm \sqrt{\beta_2^2 - \beta_1\beta_3}}{\beta_3}, \quad (5.49)$$

$$H_g^2 = \frac{\kappa_g}{6}\rho_g + \frac{\Lambda_g}{3}, \quad (5.50)$$

$$H_f^2 = \frac{\kappa_f}{6}\rho_f + \frac{\Lambda_f}{3}, \quad (5.51)$$

$$u = \frac{H_f}{H_g}, \quad (5.52)$$

and the dynamical ones

$$\dot{H}_g = -\frac{Nu\kappa_g}{4}(\rho_g + p_g), \quad (5.53)$$

$$\dot{H}_f = -\frac{N\kappa_f}{4}(\rho_f + p_f), \quad (5.54)$$

$$\frac{\dot{\xi}}{\xi} = \frac{\dot{\omega}}{\omega} = -\frac{\dot{\rho}_g}{3(\rho_g + p_g)} = -\frac{\dot{\rho}_f}{3(\rho_f + p_f)} = NH_f. \quad (5.55)$$

5.2 g-matter without f-matter

To obtain non-trivial results for the cosmology of bigravity which do not contradict the experimental evidence it is enough to study the case when f-matter is absent first considered in articles [28, 29, 30]. In general, a solution allows for an evolution starting from a matter dominated Universe and evolving to de Sitter geometry at late times. As there are many parameters β_i , $i = 0, \dots, 4$, and κ_f , the possible scenaria are diverse, including the cyclic Universe without the cosmological singularity. Even the elaborated statistical study presented, for example, in article [31] does not allow to fix the values of free parameters.

Let us suppose $\rho_f = 0 = p_f$. This case is discussed most often. As matter fields do not appear in Eq.(5.15) for the secondary constraint Ω , there are the same two branches as in the previous subsection

$$\Omega_1 = \omega H_f - \xi H_g = 0, \quad (5.56)$$

$$\Omega_2 = \beta_1 \xi^2 + 2\beta_2 \xi \omega + \beta_3 \omega^2 = 0. \quad (5.57)$$

The second branch gives nothing new, therefore we consider the first branch only. Suppose for simplicity $\kappa = \kappa_g$, then the constraints and dynamical

equations are as follows

$$\rho_g = \frac{2m^2}{\kappa_g} \left(\mu \frac{B_1(r)}{r} - B_0(r) \right), \quad (5.58)$$

$$H_g^2 = \mu \frac{m^2}{3} \frac{B_1(r)}{r}, \quad (5.59)$$

$$\dot{H}_g = \frac{N}{4} [u\kappa_g(\rho_g + p_g) + m^2(1 - ur)D_1(r)], \quad (5.60)$$

$$\dot{H}_f = -\frac{N}{4} \mu m^2 (1 - ur) \frac{D_1(r)}{r^3}, \quad (5.61)$$

$$1 - ur = \frac{(\rho_g + 3p_g) + \frac{4m^2}{\kappa_g} \left(\frac{\mu B_1}{r} - B_0 \right)}{(\rho_g + 3p_g) + \frac{2m^2}{\kappa_g} \left(B_0 - 3D_0 + \frac{3\mu D_1}{r} \right)}. \quad (5.62)$$

Here we cannot freely specify both ρ_g , and H_g as the initial data. Only one of these quantities may be fixed, and then we are to solve quartic or cubic equation for r . Any variable, besides N , can be expressed as a function of r , and dynamics of r is given as follows

$$\dot{r} = (1 - ru)NH_g, \quad (5.63)$$

whereas $N(t)$ is an arbitrary monotonic function.

For an equation of state $p_g = w\rho_g$ we obtain

$$1 - ur = \frac{(1 + w) \left(\frac{\mu B_1}{r} - B_0 \right)}{\left(\frac{1}{3} + w \right) \frac{\mu B_1}{r} - (1 + w)B_0 + \left(r + \frac{\mu}{r} \right) D_1}, \quad (5.64)$$

then the dynamical equations acquire the following form

$$\dot{r} = N \frac{(1 + w) \left(\frac{\mu B_1}{r} - B_0 \right) \sqrt{\mu \frac{m^2}{3} \frac{B_1(r)}{r}}}{\left(\frac{1}{3} + w \right) \frac{\mu B_1}{r} - (1 + w)B_0 + \left(r + \frac{\mu}{r} \right) D_1}, \quad (5.65)$$

$$\dot{H}_g = N \frac{m^2}{4} \frac{(1 + w) \left[\left(\frac{2\mu}{3r^2} + 3 \right) B_1 - \left(\frac{2\mu}{r^2} + 3 \right) r D_2 \right]}{\left(\frac{1}{3} + w \right) \frac{\mu B_1}{r} - (1 + w)B_0 + \left(\frac{\mu}{r^2} + 1 \right) r D_1}. \quad (5.66)$$

5.3 One matter minimally interacting with two metrics

Let us consider the scalar field having a minimal interaction with both $f_{\mu\nu}$ and $g_{\mu\nu}$ metrics [34, 35, 36, 37]

$$\mathcal{L}_\phi = \sqrt{-f} \left(-\frac{1}{2} f^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \mathcal{U}(\phi) \right) + \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \mathcal{U}(\phi) \right), \quad (5.67)$$

In the homogeneous cosmology we suppose $\phi = \phi(t)$ and

$$\mathbf{H}_{matter} = \mathcal{H}_\phi = N\xi^3 \rho, \quad \rho = \frac{\pi_\phi^2}{2\xi^6 \left(\frac{1}{u} + r^3\right)} + (u + r^3)\mathcal{U}(\phi), \quad (5.68)$$

Then the Hamiltonian is as follows

$$\mathbf{H} = \mathbf{H}_{potential} + \mathbf{H}_{kinetic} + \mathbf{H}_{matter} = N\mathcal{R}, \quad (5.69)$$

and the primary constraint-like equations are

$$\begin{aligned} \mathcal{R} &= \xi^3 \left[-6 \left(\frac{r^3 H_f^2}{\kappa_f} + \frac{u H_g^2}{\kappa_g} \right) + \rho(u) + \frac{2m^2}{\kappa} (B_1(r) + u B_0(r)) \right] = 0, \\ \mathcal{S} &\equiv \frac{\partial \mathcal{R}}{\partial u} = \xi^3 \left[-\frac{6H_g^2}{\kappa_g} + \rho'(u) + \frac{2m^2}{\kappa} B_0(r) \right], \end{aligned} \quad (5.70)$$

where

$$\rho'(u) = \frac{\partial \rho}{\partial u}, \quad (5.71)$$

but both \mathcal{R} and \mathcal{S} nonlinearly depend on the auxiliary variable u . Then we should solve equation $\mathcal{S} = 0$ for u and substitute the result into \mathcal{R} . Therefore this model contains only one (first class) constraint $\mathcal{R} = 0$ responsible for the time reparametrization invariance, and so it is not free of the Boulware-Deser ghost, as also the model to be considered in Section 6. Here it is possible to specify more initial data than in the previous subsections, for example, ρ , H_g , and H_f .

5.4 One matter minimally interacting with the effective metric

The new variant of interaction between the matter and bigravity is by means of the effective metric. When we do not restrict ourselves to mini-superspace the effective metric coupling leads to reappearance of the ghost as shown in Refs. [37, 38, 39, 13, 40]. Nevertheless it is claimed [37, 38] that there is a valid interesting region of applicability for this coupling at scales below the cut-off. Also, in a recent article [41] it is stated that the ghost can be excluded in a version of the terad formalism that is non-equivalent to the metric one.

The first analysis of cosmology in this model was given in article [42]. The effective metric is constructed according to the formula

$$\mathcal{G}_{\mu\nu} = \alpha^2 g_{\mu\nu} + 2\alpha\beta g_{\mu\alpha} \mathbf{X}_\nu^\alpha + \beta^2 f_{\mu\nu}, \quad (5.72)$$

in tetrad language this corresponds to exploiting the linear combination of the two tetrads:

$$\alpha E_\mu^A + \beta F_\mu^A, \quad (5.73)$$

such a possibility for the special case $\alpha = \beta$ has been mentioned first in Ref. [25]. In minisuperspace the components of the effective metric will be denoted as follows

$$\mathcal{G}_{00} = -\mathcal{N}^2, \quad \mathcal{G}_{0i} = 0, \quad \mathcal{G}_{ij} = a^2 \delta_{ij} \equiv \psi_{ij}, \quad (5.74)$$

where

$$\mathcal{N} = N(\alpha u + \beta), \quad a = \alpha \xi + \beta \omega. \quad (5.75)$$

There are the following standard relations:

$$\sqrt{-\mathcal{G}} = \mathcal{N} \sqrt{\psi}, \quad \sqrt{\psi} = a^3, \quad (5.76)$$

where

$$\mathcal{G} = \det(\mathcal{G}_{\mu\nu}), \quad \psi = \det(\psi_{ij}). \quad (5.77)$$

The primary constraints are

$$\mathcal{S} = -\frac{6\xi^3}{\kappa_g} H_g^2 + \alpha \hat{\mathcal{H}}^{(m)} + \frac{2m^2}{\kappa} \xi^3 B_0(r), \quad (5.78)$$

$$\mathcal{R}' = -\frac{6\omega^3}{\kappa_f} H_f^2 + \beta \hat{\mathcal{H}}^{(m)} + \frac{2m^2}{\kappa} \xi^3 B_1(r), \quad (5.79)$$

where it is supposed that the scalar field canonical variables (ϕ, π_ϕ) depend on time variable only and therefore

$$\hat{\mathcal{H}}^{(m)} = \frac{\pi_\phi^2}{2a^3} + a^3 \mathcal{U}(\phi). \quad (5.80)$$

We introduce the following notation for the matter energy density

$$\rho = \frac{\hat{\mathcal{H}}^{(m)}}{a^3} \equiv \frac{\pi_\phi^2}{2a^6} + \mathcal{U}(\phi). \quad (5.81)$$

The minisuperspace Hamiltonian is as follows

$$H = N(\mathcal{R}' + u\mathcal{S}), \quad (5.82)$$

and the Poisson brackets are

$$\{\xi, H_g\} = -\frac{\kappa_g}{12\xi^2}, \quad \{\omega, H_f\} = -\frac{\kappa_f}{12\omega^2}, \quad \{\phi, \pi_\phi\} = 1. \quad (5.83)$$

It is suitable to write Eqs.(5.78), (5.79) in a different form

$$\mathcal{S} = \xi^3 \left[-\frac{6}{\kappa_g} H_g^2 + \alpha(\alpha + \beta r)^3 \rho + \frac{2m^2}{\kappa} B_0(r) \right], \quad (5.84)$$

$$\mathcal{R}' = \omega^3 \left[-\frac{6}{\kappa_f} H_f^2 + \frac{\beta}{r^3} (\alpha + \beta r)^3 \rho + \frac{2m^2}{\kappa} \frac{B_1(r)}{r^3} \right]. \quad (5.85)$$

Demanding the fulfilment of equation $\mathcal{S} = 0$ in the course of evolution we obtain the secondary constraint,

$$\dot{\mathcal{S}} = \{\mathcal{S}, H\} = N\Omega, \quad \rightarrow \quad \Omega = 0, \quad (5.86)$$

where Ω has the following form

$$\Omega = \{\mathcal{S}, \mathcal{R}'\} = \frac{6m^2}{\kappa} (\omega H_f - \xi H_g) \left(\beta_1 \xi^2 + 2\beta_2 \xi \omega + \beta_3 \omega^2 - \frac{\kappa}{2m^2} \alpha \beta a^2 p \right), \quad (5.87)$$

and we have introduced the notation for pressure

$$p = \frac{\pi_\phi^2}{2a^6} - \mathcal{U}(\phi). \quad (5.88)$$

After calculating

$$\{\Omega, \mathcal{S}\} = \Delta \neq 0, \quad (5.89)$$

we see that Ω and \mathcal{S} are 2nd class constraints. It is possible to introduce the Dirac brackets

$$\{F, G\}_D = \{F, G\} - \frac{\{F, \Omega\}\{\mathcal{S}, G\} - \{F, \mathcal{S}\}\{\Omega, G\}}{\Delta}, \quad (5.90)$$

and consider \mathcal{R}' as a first class constraint. The reduced Hamiltonian and equations of motion will be as follows

$$H_{\text{reduced}} = N\mathcal{R}', \quad \dot{F} = \{F, H_{\text{reduced}}\}_D. \quad (5.91)$$

An equivalent way to derive the Hamiltonian equations is to exploit the Poisson brackets, but after their calculation insert a solution for the variable u determined from the equation

$$\dot{\Omega} = \{\Omega, H\} \equiv N(\{\Omega, \mathcal{R}'\} + u\{\Omega, \mathcal{S}\}) = 0, \quad (5.92)$$

i.e.

$$u = -\frac{1}{\Delta}\{\Omega, \mathcal{R}'\}. \quad (5.93)$$

As Ω is factorized (5.87)(compare Eq.(5.16))

$$\Omega = \Omega_1 \Omega_2, \quad (5.94)$$

there are two different solutions for it:

$$\Omega_1 = \omega H_f - \xi H_g = 0, \quad (5.95)$$

or

$$\Omega_2 = \beta_1 \xi^2 + 2\beta_2 \omega \xi + \beta_3 \omega^2 - \frac{\kappa}{2m^2} \alpha \beta a^2 p = 0. \quad (5.96)$$

Below we will consider both the two cases.

With the Hamiltonian (5.82) and the Poisson brackets (5.83) we can derive the kinematical Hamiltonian equations, which, of course, are equivalent to the definitions of the Hubble constants (see (4.23) to compare) and the scalar field momentum

$$\dot{\xi} = \{\xi, H_g\} N \left(\frac{\partial \mathcal{R}'}{\partial H_g} + u \frac{\partial \mathcal{S}}{\partial H_g} \right) = Nu \xi H_g, \quad (5.97)$$

$$\dot{\omega} = \{\omega, H_f\} N \left(\frac{\partial \mathcal{R}'}{\partial H_f} + u \frac{\partial \mathcal{S}}{\partial H_f} \right) = N\omega H_f, \quad (5.98)$$

$$\dot{\phi} = \{\phi, H\} = N \left(\frac{\partial \mathcal{R}'}{\partial \pi_\phi} + u \frac{\partial \mathcal{S}}{\partial \pi_\phi} \right) = N(\alpha u + \beta) \frac{\pi_\phi}{a^3}. \quad (5.99)$$

Combining the kinematical and dynamical Hamiltonian equations for the scalar field (i.e. for the matter)

$$\dot{\phi} = N(\alpha u + \beta) \frac{\pi_\phi}{a^3}, \quad \dot{\pi}_\phi = -N(\alpha u + \beta) a^3 \mathcal{U}'(\phi)$$

we obtain the energy conservation law

$$\frac{\pi_\phi \dot{\pi}_\phi}{a^3} + a^3 \dot{\phi} \mathcal{U}'(\phi) = 0,$$

which can be written in the standard form

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \quad (5.100)$$

Dynamical Hamiltonian equations for the gravitational variables are the following

$$\dot{H}_f = - \frac{N\kappa_f}{4r^3} \left[\beta (\alpha + \beta r)^3 \left(\rho + p \frac{\beta + \alpha u}{\beta + \frac{\alpha}{r}} \right) + \right. \quad (5.101)$$

$$\left. + \frac{2m^2}{\kappa} (1 - ur) D_1(r) \right], \quad (5.102)$$

$$\dot{H}_g = - \frac{Nu\kappa_g}{4} \left[\alpha (\alpha + \beta r)^3 \left(\rho + \frac{p}{ur} \frac{\beta + \alpha u}{\beta + \frac{\alpha}{r}} \right) - \right. \quad (5.103)$$

$$\left. - \frac{2m^2}{\kappa} (1 - ur) \frac{D_1(r)}{u} \right]. \quad (5.104)$$

(see (5.23), (5.24) for comparison). Normally the matter equation of state is known also.

5.4.1 First branch

Here we consider the following solution of the constraint $\Omega = 0$:

$$\Omega_1 = \omega H_f - \xi H_g = 0, \quad \rightarrow \quad H_f = r^{-1} H_g.$$

Then for the Hubble parameters

$$H_f = \frac{\dot{\omega}}{N\omega}, \quad H_g = \frac{\dot{\xi}}{Nu\xi}, \quad H = \frac{\dot{a}}{N(\alpha u + \beta)a}, \quad (5.105)$$

the following relations are valid

$$H_g = r H_f, \quad H_f = H \left(\frac{\alpha}{r} + \beta \right), \quad H_g = H(\alpha + \beta r). \quad (5.106)$$

Constraints (5.84), (5.85) are equivalent to equations of the Friedmann form

$$H_f^2 = \frac{\kappa_f}{6r^3} \left(\beta (\alpha + \beta r)^3 \rho + \frac{2m^2}{\kappa} B_1(r) \right), \quad (5.107)$$

$$H_g^2 = \frac{\kappa_g}{6} \left(\alpha (\alpha + \beta r)^3 \rho + \frac{2m^2}{\kappa} B_0(r) \right), \quad (5.108)$$

with a lot of new parameters $\alpha, \beta, \beta_0, \dots, \beta_4$ and a new variable $r(t)$ involved. It is suitable to rewrite these equations as follows

$$H_f^2 = \frac{\kappa_f}{6}\rho_f + \frac{\Lambda_f}{3}, \quad (5.109)$$

$$H_g^2 = \frac{\kappa_g}{6}\rho_g + \frac{\Lambda_g}{3}, \quad (5.110)$$

where

$$\rho_f = \beta \left(\frac{\alpha}{r} + \beta \right)^3 \rho, \quad (5.111)$$

$$\rho_g = \alpha(\alpha + \beta r)^3 \rho, \quad (5.112)$$

$$\Lambda_f = m^2 \frac{\kappa_f}{\kappa} \frac{B_1(r)}{r^3}, \quad (5.113)$$

$$\Lambda_g = m^2 \frac{\kappa_g}{\kappa} B_0(r). \quad (5.114)$$

From Eq.(5.93) we obtain

$$u = \frac{\frac{B_1}{r} - 3D_2 + 3\mu^{-1}rD_1 + \frac{\kappa}{2m^2}\beta(\alpha + \beta r)^3 \left(\rho r^{-1} + 3p \frac{\beta - \mu^{-1}r\alpha}{\alpha + \beta r} \right)}{3D_1 + \mu^{-1}r(B_0 - 3D_0) + \frac{\kappa}{2m^2}\alpha(\alpha + \beta r)^3 \left(\rho \mu^{-1}r - 3p \frac{\beta - \mu^{-1}r\alpha}{\alpha + \beta r} \right)}, \quad (5.115)$$

where

$$D_i = \beta_i + 2\beta_{i+1}r + \beta_{i+2}r^2, \quad B_i = D_i + rD_{i+1}. \quad (5.116)$$

With account for Eqs.(5.106) we can reformulate our problem to a couple of Friedmann-like equations for H

$$H^2 = \frac{\kappa_g}{6}\alpha(\alpha + \beta r)\rho + \frac{m^2}{3} \frac{\kappa_g}{\kappa} \frac{B_0}{(\alpha + \beta r)^2}, \quad (5.117)$$

$$H^2 = \frac{\kappa_f}{6r}\beta(\alpha + \beta r)\rho + \frac{m^2}{3r} \frac{\kappa_f}{\kappa} \frac{B_1}{(\alpha + \beta r)^2}, \quad (5.118)$$

This system can be solved for the matter density

$$\rho = \mu^{-1}r \frac{2m^2}{\kappa} \frac{\frac{\mu B_1(r)}{r} - B_0(r)}{(\alpha + \beta r)^3 (\mu^{-1}r\alpha - \beta)}. \quad (5.119)$$

Vice versa it is possible to exclude ρ , and obtain

$$H^2 = \frac{m^2}{3} \frac{\alpha B_1(r) - \beta B_0(r)}{(\alpha + \beta r)^2} \frac{\kappa_g}{(\mu^{-1} r \alpha - \beta) \kappa}, \quad (5.120)$$

or we can express the cosmological term as a function of r :

$$\Lambda = m^2 \frac{\kappa_g}{\kappa} \frac{B_0}{(\alpha + \beta r)^2}. \quad (5.121)$$

When the unobservable variable r is given, we can calculate ρ , H , and Λ from Eqs.(5.119)–(5.121). Vice versa, if we take an initial value for ρ or H , after solving a quartic or cubic equation we can find a few solutions for r . The dynamical equation for r

$$\dot{r} = \dot{a}(1 - ur) \frac{\alpha + \beta r}{\beta + \alpha u}, \quad (5.122)$$

is sufficient to predict its evolution, and therefore the evolution (forward or backward in time) of all observables. For the important factor $1 - ur$ we get the following expression

$$1 - ur = \frac{\alpha(\alpha + \beta r)^3(\rho + 3p) \left(1 - \frac{\beta\mu}{\alpha r}\right) + \frac{4m^2}{\kappa} \left(\frac{\mu B_1}{r} - B_0\right)}{\alpha(\alpha + \beta r)^3 \left(\rho - 3p \frac{\beta - \mu^{-1} r \alpha}{\alpha + \beta r}\right) + \frac{2m^2}{\kappa} \left(B_0 - 3D_0 + \frac{3\mu D_1}{r}\right)}. \quad (5.123)$$

Eq.(5.123) may be simplified if the matter has the equation of state $p = w\rho$:

$$1 - ur = \frac{3(1 + w) \left(\frac{\mu B_1}{r} - B_0\right)}{B_0 - 3D_0 + \frac{3\mu D_1}{r} + \left(\frac{\mu B_1}{r} - B_0\right) \left(\frac{1}{1 - \frac{\mu\beta}{\alpha r}} + \frac{3w}{1 + \frac{\beta}{\alpha r}}\right)}. \quad (5.124)$$

$r(t)$ may be non-monotonic and may change its behavior at those points where $r = 1/u$.

5.4.2 Second branch

The second branch $\Omega_2 = 0$ gives the following equation

$$\beta_1 \xi^2 + 2\beta_2 \xi \omega + \beta_3 \omega^2 - \frac{\kappa}{2m^2} \alpha \beta a^2 p = 0,$$

or, equivalently,

$$p = \frac{2m^2}{\kappa} \frac{D_1(r)}{\alpha\beta(\alpha + \beta r)^2}.$$

This equation gives a possibility to determine the pressure of matter p , and also the density ρ , through the matter equation of state, as a function of variable r . Then from constraints $\mathcal{S} = 0$, $\mathcal{R}' = 0$ we can find H_g and H_f . Evolution of these variables are provided by the Hamiltonian equations

$$\dot{H}_g = -\frac{Nu\kappa_g}{4}\alpha(\alpha + \beta r)^3(\rho + p), \quad (5.125)$$

$$\dot{H}_f = -\frac{N\kappa_f}{4r^3}\beta(\alpha + \beta r)^3(\rho + p). \quad (5.126)$$

The combination of Eqs.(5.125), (5.126) gives the following relation

$$\frac{\dot{H}_g}{\dot{H}_f} = \mu^{-1} \frac{\alpha}{\beta} u r^3. \quad (5.127)$$

6 Non-dRGT bigravity

Let the potential in bigravity will be of non-dRGT type, for example, as taken from the Relativistic Theory of Gravitation (RTG) [43]:

$$U = \frac{1}{2} \left[\sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} f_{\mu\nu} - 1 \right) - \sqrt{-f} \right]. \quad (6.1)$$

Then in 3 + 1 minisuperspace notations

$$\sqrt{-g} = Nu\xi^3, \quad \sqrt{-f} = N\omega^3, \quad g^{\mu\nu} f_{\mu\nu} = \frac{1}{u^2} + 3r^2, \quad (6.2)$$

$$U = \frac{1}{2} N \left(\frac{\xi^3}{2u} + \frac{3}{2} u \xi \omega^2 - u \xi^3 - \omega^3 \right) \equiv N\tilde{U}, \quad (6.3)$$

and we obtain

$$V = \frac{\partial \tilde{U}}{\partial u} = \frac{1}{2} \left(-\frac{\xi^3}{2u^2} - \xi^3 + \frac{3}{2} \xi \omega^2 \right), \quad (6.4)$$

$$W = \tilde{U} - u \frac{\partial \tilde{U}}{\partial u} = \frac{1}{2} \left(\frac{\xi^3}{u} - \omega^3 \right). \quad (6.5)$$

Therefore constraint-like equations are the following

$$\mathcal{R}' = \mathcal{H} + \frac{2m^2}{\kappa}W = \omega^3 \left[-\frac{6H_f^2}{\kappa_f} + \rho_f + \frac{m^2}{\kappa} \left(\frac{1}{ur^3} - 1 \right) \right] = 0, \quad (6.6)$$

$$\mathcal{S} = \bar{\mathcal{H}} + \frac{2m^2}{\kappa}V = \xi^3 \left[-\frac{6H_g^2}{\kappa_g} + \rho_g + \frac{m^2}{\kappa} \left(-\frac{1}{2u^2} - 1 + \frac{3}{2}r^2 \right) \right] = 0. \quad (6.7)$$

but in fact both Eqs.(6.6), (6.7) contain auxiliary variable u . Then we should solve one of these equations for u and substitute the solution into another equation. So, there is only one constraint in fact. It is a first class constraint arising as a result of the time reparametrisation invariance having place in the bi-RTG. For this reason the variable N stays arbitrary. One can take, for example, ρ_g, ρ_f, H_g , and H_f as the initial data. We see that this theory has one more gravitational degree of freedom than the bigravity with the dRGT potential. As a result, it is not suitable for the description of the observable cosmology, and so never has been discussed in the literature.

The dynamical Hamiltonian equations here are as follows

$$\dot{H}_f = -\frac{N\kappa_f}{4} \left[\rho_f + p_f + \frac{m^2}{\kappa} \frac{1 - (ur)^2}{ur^3} \right], \quad (6.8)$$

$$\dot{H}_g = -\frac{Nu\kappa_g}{4} \left[\rho_g + p_g - \frac{m^2}{\kappa} \frac{1 - (ur)^2}{u^2} \right]. \quad (6.9)$$

6.1 Non-dRGT massive gravity

When we take one metric, for example, $f_{\mu\nu}$, as a background Minkowskian one, we put $\omega \equiv 1$, $H_f = 0$, $\rho_f = 0$, and the solution of Eq.(6.6) is

$$u = \frac{1}{r^3} \equiv \frac{\xi^3}{\omega^3} = \xi^3. \quad (6.10)$$

After substitution of this into \mathcal{S} we obtain the Friedmann equation in the RTG:

$$H_g^2 = \frac{\kappa_g \rho_g}{6} - \frac{m^2}{12\xi^6} \frac{\kappa_g}{\kappa} (1 + 2\xi^6 - 3\xi^4). \quad (6.11)$$

If we take $\kappa = \kappa_g = 16\pi G$, then it is possible to rewrite Eq.(6.11) as follows

$$H_g^2 = \frac{8\pi G}{3} \rho_g + \frac{\Lambda}{3}, \quad (6.12)$$

where

$$\Lambda(\xi) = -\frac{m^2}{2} \left(\frac{1}{2\xi^6} + 1 - \frac{3}{2\xi^2} \right). \quad (6.13)$$

We can take ρ_g and H_g as the initial data. Then $\Lambda(\xi)$ is determined from Eq.(6.12), and ξ could be found as a solution of the bi-cubic equation (6.13). In contrast to the massive gravity with dRGT potential this theory admits the homogeneous and isotropic dynamical Friedmann cosmology. It is due to the fact that the gravitational field has one additional degree of freedom here. But this degree of freedom is unfortunately a ghost. The effective cosmological constant is negative, and therefore any expansion should in future be transformed into contraction. The fate of the Universe is cyclic in this model. Of course, a quintessence matter field may provide an accelerated expansion for a finite interval of time.

7 Conclusion

The cosmology of bigravity is an open problem that is under active study [6, 7, 8, 27]. This article is written in order to demonstrate the beauty and power of the Hamiltonian formalism in this field of research. We would like to mention that it is a direct road to develop the quantum cosmology in bigravity.

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8 Appendix

The straightforward calculation of traces for matrix \mathbf{X} starting from Eq.(3.12) gives the following results

$$\text{Tr}\mathbf{X} = -A + D, \quad (8.1)$$

$$\text{Tr}\mathbf{X}^2 = A^2 - 2(BC) + \text{Tr}D^2, \quad (8.2)$$

$$\text{Tr}\mathbf{X}^3 = -A^3 + 3A(BC) - (BDC) - \text{Tr}D^3. \quad (8.3)$$

Then it is easy to estimate the function \tilde{U} and its derivatives

$$\begin{aligned}
V &= \beta_0 e + \beta_1 e \left(x + \frac{y}{\varepsilon + 1} \right) + \beta_2 e \left[x^2 - \text{Tr}x^2 + 2 \frac{xy - (px^2p)}{\varepsilon + 1} \right] + \\
&+ \beta_3 e \left[x^3 - 3x \text{Tr}x^2 + 2 \text{Tr}x^3 + \frac{6(px^3p) - 6x(px^2p) - 3y \text{Tr}x^2}{\varepsilon + 1} \right] \\
V_i &= -\beta_1 e (fp)^i + 2\beta_2 e [(fx^T p)^i - (fp)^i x] + \\
&+ 3\beta_3 e [2x(fx^T p)^i + \text{Tr}x^2 (fp)^i - 2(f(x^2)^T p)^i] = -\mathbf{f}_{ia} C_{ab} p_b, \\
W &= \beta_4 f + \beta_1 e \varepsilon + 2\beta_2 e \varepsilon \left(x - \frac{y}{\varepsilon(\varepsilon + 1)} \right) + \\
&+ \beta_3 e \left[x^3 - 3x \text{Tr}x^2 + 2 \text{Tr}x^3 + \frac{6(px^3p) - 6x(px^2p) - 3z \text{Tr}x^2}{\varepsilon + 1} \right],
\end{aligned}$$

where

$$C_{ab} = e [\delta_{ab}(\beta_1 + 2\beta_2 x - 3\beta_3 \text{Tr}x^2) - 2x_{ba}(\beta_2 + 3\beta_3 x) + 6\beta_3 x_{bc} x_{ca}]. \quad (8.4)$$

The following notations were used above

$$x_{ab} = \mathbf{f}_{ia} \mathbf{e}_b^i, \quad (8.5)$$

$$x = x_{aa}, \quad (8.6)$$

$$\text{Tr}x^2 = x_{ab} x_{ba}, \quad (8.7)$$

$$\text{Tr}x^3 = x_{ab} x_{bc} x_{ca}, \quad (8.8)$$

$$y_{ab} = p_a p_c x_{cb}, \quad (8.9)$$

$$y \equiv (pxp) = p_a x_{ab} p_b = y_{aa}, \quad (8.10)$$

$$(px^2p) = p_a x_{ab} x_{bc} p_c, \quad (8.11)$$

$$(px^3p) = p_a x_{ab} x_{bc} x_{cd} p_d, \quad (8.12)$$

$$(ufp) = u^i f_{ia} p^a, \quad (8.13)$$

$$(ufx^T p) = u^i f_{ia} x_{ba} p_b, \quad (8.14)$$

$$(uf(x^2)^T p) = u^i f_{ia} x_{ba} x_{cb} p_c. \quad (8.15)$$

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