ISSN 0027-1349, Moscow University Physics Bulletin, 2010, Vol. 65, No. 6, pp. 539–542. © Allerton Press, Inc., 2010.
Original Russian Text © V.Ch. Zhukovsky, K.G. Klimenko, I.E. Frolov, 2010, published in Vestnik Moskovskogo Universiteta. Fizika, 2010, No. 6, pp. 106–109.

Density Waves in Quark Matter within the Nambu–Jona–Lasinio Model in an External Magnetic Field

V. Ch. Zhukovsky[a] *, K. G. Klimenko[b] **, and I. E. Frolov[a]

[a] *Faculty of Physics, Moscow State University, Moscow, 119991 Russia*
[b] *Institute for High Energy Physics, ul. Pobedy, d. 1, Protvino, Moscow oblast, 142281 Russia*
*e-mail: *zhukovsk@phys.msu.ru, **kklim@ihep.ru*

Received: July 3, 2010

Abstract—The possibility of forming static dual scalar and pseudo-scalar density wave condensates in dense quark matter is considered for the Nambu–Jona–Lasinio model in an external magnetic field. Within a mean-field approximation, the effective potential of the theory is obtained and its extrema are numerically studied and a phase diagram of the system is constructed. It is shown that the presence of a magnetic field favors the formation of spatially inhomogeneous condensate configurations at low temperatures.

Keywords: chiral condensate, pion condensate, density waves, magnetic catalysis.

DOI: 10.3103/S0027134910060202

INTRODUCTION

One of the most common effective theories of quantum chromodynamics is the Nambu–Jona–Lasinio (NJL) model [1]. It can explain many of the properties of quarks and light mesons based on the mechanism of dynamical chiral symmetry breaking and the formation of chiral and pion condensates under.

In the study of physical phenomena in the framework of the NJL model it is assumed, as a rule, that vacuum condensates are homogeneous and isotropic in space. This assumption, however, cannot be justified for medium and high densities of matter and coupling constants [2]. The possibility of forming static coherent density waves of quarks–antiquarks and pion condensates (Dual-chiral density waves) [3] of the form

$$\langle \bar{\psi}\psi \rangle = \Delta \cos \mathbf{qr},$$
$$\langle \bar{\psi} i\gamma^5 \tau_3 \psi \rangle = \Delta \sin \mathbf{qr}, \qquad (1)$$
$$\langle \bar{\psi} i\gamma^5 \tau_{1,2} \psi \rangle = 0,$$

where Δ is the amplitude of the condensate, $\mathbf{q}$ is a fixed wave vector, τ_i are Pauli matrices, while $\langle \bar{\psi}\psi \rangle^2 + \langle \bar{\psi} i\gamma^5 \boldsymbol{\tau} \psi \rangle^2 = \Delta^2$ at any point in space has been considered in the literature. It has been shown that a density wave in a quark medium can be formed at low temperatures in the transition region between the massive and massless phases of the NJL model. The behavior in external fields, however, has not been investigated.

In this paper we consider the possibility of a configuration of condensates of this type in the presence of an external magnetic field.

1. PHYSICAL MODEL

We will explore the model used in [3], given $N_f = 2$ varieties of quarks (u and d) with zero current mass and $N_c = 3$ colors. For the configuration of vacuum condensates (1) the NJL Lagrangian in the mean field approximation has the form:

$$\mathcal{L} = \bar{\psi}(i\gamma D + \mu\gamma^0 - m(\cos \mathbf{qr} + i\gamma^5\tau_3 \sin \mathbf{qr}))\psi - \frac{m^2}{4G}, \qquad (2)$$

where $m = -2G\Delta$, G is the coupling constant, μ is the chemical potential, $D = \partial + iQeA$, Q is the operator of the electric charge (with eigenvalues $Q_u = 2/3$ and $Q_d = -1/3$), A is the potential of the electromagnetic field, and $e > 0$. We assume that the wave vector $\mathbf{q}$ and the magnetic field vector $\mathbf{H}$ are directed along one axis, the z-axis.

It is easy to see that the transformation of the fields $\psi \longrightarrow e^{i\gamma^5\tau_3 bx}\psi$, $\bar{\psi} \longrightarrow \bar{\psi} e^{i\gamma^5\tau_3 bx}$, where $b = q/2$, $q^\mu \equiv (0, \mathbf{q})$, $x^\mu \equiv (t, \mathbf{r})$ leads (2) to the form:

$$\mathcal{L} = \bar{\psi}(i\gamma D + \mu\gamma^0 - m + \gamma^5\tau_3\gamma b)\psi - \frac{m^2}{4G}, \qquad (3)$$

where the measure of the path integral $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ remains invariant, which is easy to verify using the method of Fujikawa [4]. The term $\bar{\psi}\gamma^5\tau_3\gamma b\psi$ in (3) represents the background CPT-odd axial-vector interaction of fermions.

EFFECTIVE ACTION

It is easy to see that the theory described by the Lagrangian (3) in the single-loop approximation of the problem is formally reduced to finding an effective action for one electron (with the corresponding replacement of the electric charge of the particle e on $|Q_u|e$ and $|Q_d|e$; in this case we consider a negatively charged particle):

$$\Gamma^{(1)} = \int d^4x\left(-\frac{m^2}{4G}\right) + N_c\Gamma_{\tilde{e}}^{(1)}\Big|_{e\to|Q_u|e} + N_c\Gamma_{\tilde{e}}^{(1)}\Big|_{e\to|Q_d|e}, \quad (4)$$

$$\begin{aligned}\Gamma_{\tilde{e}}^{(1)} &= \frac{1}{i}\ln\mathrm{Det}(i\gamma D + \mu\gamma^0 - m - \gamma^5\gamma b)\\ &= \frac{1}{i}\ln\mathrm{Det}(i\partial^0 + \mu - H_\mathrm{D}),\end{aligned} \quad (5)$$

where $H_\mathrm{D} = \boldsymbol{\alpha}\mathbf{P} + \gamma^0 m - \Sigma_3 b$ is a modified Dirac Hamiltonian and $\mathbf{P} = \mathbf{p} + e\mathbf{A}$; here and in the following $b^\mu = (0, 0, 0, b)$, $\mathbf{H} = (0, 0, H)$, $H > 0$. To calculate the expression (5) we need the eigenfunctions and the spectrum of the Hamiltonian H_D.

Let $\mathbf{A} = (0, Hx, 0)$. Since $[\Sigma_3, (\boldsymbol{\alpha}\mathbf{P})^2] = 0$, the eigenfunctions H_D have a standard form (see [5]):

$$\Psi_{nqp}(x, y, z) = \frac{1}{\sqrt{2\pi}}e^{ipz}\frac{1}{\sqrt{2\pi}}e^{iqy}\begin{pmatrix} c_1u_{n-1}(\xi)\\ ic_2u_n(\xi)\\ c_3u_{n-1}(\xi)\\ ic_4u_n(\xi)\end{pmatrix}(eH)^{\frac{1}{4}},$$

where $u_n(\xi)$ are Hermite functions and $\xi = \sqrt{eH}x + \frac{q}{\sqrt{eH}}$, $\{c_i\}$ are spin coefficients. For each fixed $n > 0$, we have the eigenvalue problem for the matrix $K\,(4\times 4)$ and column $\{c_i\}$, where

$$K = \alpha_1 p_\perp + \alpha_3 p + \gamma^0 m - \Sigma_3 b, \quad p_\perp = \sqrt{2eHn}.$$

We perform an unitary transformation: $\tilde{K} = U^{-1}KU$, where $U = e^{i\Sigma_2\frac{\pi}{2}}e^{i\gamma^0\Sigma_2\frac{\pi}{2}}$, then

$$\tilde{K} = \alpha_1\tilde{p}_\perp + \alpha_3\tilde{p} + \gamma^0 m + \gamma^0\Sigma_3\tilde{\mu}H,$$
$$\tilde{p}_\perp = p, \quad \tilde{p} = -p_\perp, \quad \tilde{\mu}H = b.$$

The matrix $\tilde{K}$ corresponds to the problem of motion of an electron with anomalous magnetic moment $\tilde{\mu}$; the solutions of this problem are known [6][1]. The case $n = 0$, however, requires special consideration in an explicit form; this can be easily done.

The final expression for the spectrum has the form:

$$E_{np\zeta\epsilon} = \begin{cases}\epsilon\sqrt{(\zeta(m^2+p^2)+b)^2 + 2eHn}, & n > 0,\\ \epsilon\sqrt{m^2+p^2} + b, & n = 0,\end{cases} \quad (6)$$

where $n = 0, 1, \ldots, -\infty < p < \infty$, $\zeta = \pm 1$ and $\epsilon = \pm 1$. For $n = 0$, the spin quantum number ζ is not part of the spectrum, in addition, there is always the degeneracy in q $(-\infty < p < \infty)$. Note that the asymmetry of the spectrum of particles and antiparticles at $n = 0$ is related to the CPT odd of the interaction considered and the existence of a preferred direction $\mathbf{H}$.

Knowing the explicit form of wave functions, we can find the perturbation correction to the energy $\Delta E_\perp$, due to the deviation of $\mathbf{b}$ from the direction $\mathbf{H}$. Let $\mathbf{b} = (b_\perp, 0, b)$. Omitting the intermediate calculations, we present the corresponding asymptotic expression (constructed with account for possible intersection of the near unperturbed levels):

$$\Delta E_\perp = \sum_\pm (f_\pm|_{\epsilon'=\epsilon} + g_\pm|_{\epsilon'=-\epsilon})\big|_{n'=n\pm 1,\,\zeta'=\zeta},$$

where

$$f_\pm = \frac{1}{2}(-E + E' + \mathrm{sgn}_\pm(E-E')\sqrt{(E-E')^2 + 4D_\pm}),$$

$$g_\pm = \frac{D_\pm}{E-E'},$$

$$D_\pm = 2b_\perp^2\left(1 \mp \zeta'\epsilon'\sqrt{1-\frac{2eHn'}{E'^2}}\right)\left(1 \pm \zeta\epsilon\sqrt{1-\frac{2eHn}{E^2}}\right),$$

and we introduce the notation: $E \equiv E_{np\zeta\epsilon}$, $E' \equiv E_{n'p'\zeta'\epsilon'}$; the function $\mathrm{sgn}_\pm(x)$ returns the sign of its argument for $x \neq 0$ and is subject to the condition $\mathrm{sgn}_\pm(0) = \pm 1$. Its use is explained by the fact that we want to save the meaning of a small correction to the initial energy level E with a certain set of introduced quantum numbers $np\zeta\epsilon$ (at least away from the intersection points of the unperturbed levels) for $\Delta E_\perp$.

After introducing the normalization 4-volume $L_tL_xL_yL_z$ we can now calculate (5):

$$\begin{aligned}\Gamma_{\tilde{e}}^{(1)} &= \frac{1}{i}\ln\mathrm{Det}(i\partial^0 + \mu - H_\mathrm{D})\\ &= \frac{1}{2i}\mathrm{Tr}\ln(-(i\partial^0)^2 + (H_\mathrm{D}-\mu)^2)\\ &= \frac{1}{2i}\int dp^0\sum_{(n)}\int d^4x\frac{1}{\sqrt{2\pi}}\\ &\times e^{ip^0t}\Psi^+\ln(-(i\partial^0)^2 + (H_\mathrm{D}-\mu)^2)\frac{1}{\sqrt{2\pi}}e^{-ip^0t}\Psi\end{aligned}$$

[1] Note that the form of spin coefficients does not depend on the choice of the gauge potential $\mathbf{A}$.

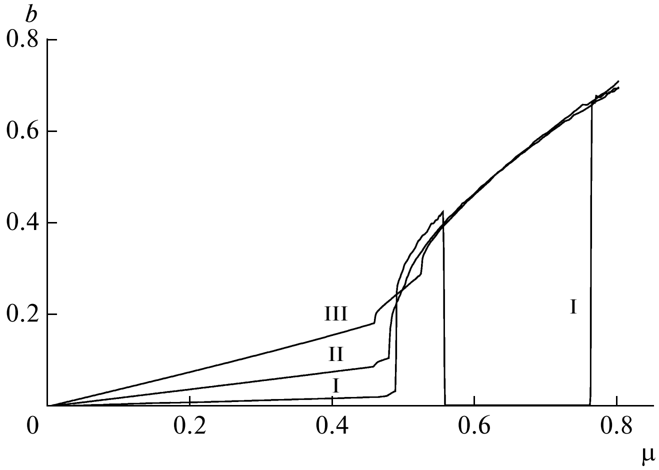

Fig. 1. Dependence of the parameter b on the chemical potential μ at $G = 6$ and $T = 0$ for $\sqrt{eH} = 0.15$ (I), $\sqrt{eH} = 0.3$ (II) and $\sqrt{eH} = 0.45$ (III) (all quantities are dimensionless). In graph I, a site with $b = 0$, which corresponds to the symmetric phase with $m = 0$, was observed.

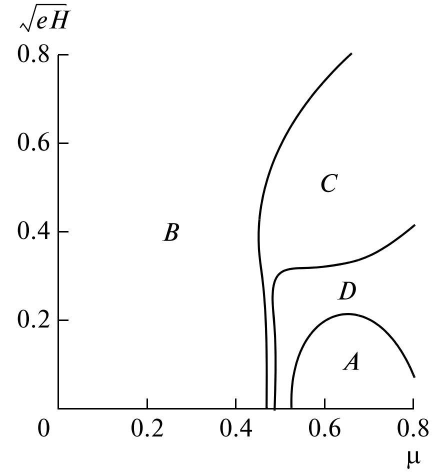

Fig. 2. A phase diagram of quark matter at $G = 6$ and $T = 0$. The parameters μ and $\sqrt{eH}$ are dimensionless. Along with the symmetric phase A and the massive phases B and C, the noted phase D with a strong spatial condensate inhomogeneity occurs at $H \longrightarrow 0$.

$$= \frac{1}{2i}\int dp^0 \sum_{(n)} \ln(-(p^0)^2 + (E-\mu)^2)\frac{L_t}{2\pi}\frac{L_z}{2\pi}\frac{L_y}{2\pi}.$$

In the last expression (n) denotes the set of all quantum numbers and

$$\sum_{(n)} = \sum_{n\zeta\epsilon}\int dp\int dq = \sum_{n\zeta\epsilon}\int dp eHL_x,$$

as $q = -x_0 eH$, where x_0 is the coordinate of the center of the wave function Ψ on the x axis. To obtain the thermodynamic potential of the system at finite temperature T, we apply the technique of Matsubara [7]:

$$\int_{-\infty}^{\infty}\frac{dp^0}{2\pi} \longrightarrow \int_{-i\infty}^{i\infty}\frac{dp^0}{2\pi} = i\int_{-\infty}^{\infty}\frac{dp^4}{2\pi} \longrightarrow i\frac{1}{\beta}\sum_{k=-\infty}^{+\infty},$$

while $p^4 \longrightarrow \omega_k = 2\pi/\beta(k + 1/2)$, where $\beta = 1/T$; the sum over k can be derived analytically.

Finally, for the thermodynamic potential $\Omega = -\frac{\Gamma^{(1)}}{L_t L_x L_y L_z}$, we obtain:

$$\Omega = \frac{m^2}{4G} + N_c\Omega_{\bar{e}}\big|_{e\to|Q_u|e} + N_c\Omega_{\bar{e}}\big|_{e\to|Q_d|e}, \quad (7)$$

$$\Omega_{\bar{e}} = -\frac{1}{2}\frac{eH}{(2\pi)^2}\int dp\sum_{n\zeta\epsilon}\left(|E-\mu| + \frac{2}{\beta}\ln(1+e^{-\beta|E-\mu|})\right). \quad (8)$$

Expression (8) contains a divergent vacuum contribution (corresponding to $\mu = 0$, $T = 0$). For its regularization, we use the proper time method [8]:

$$\Omega_v = -\frac{1}{2}\frac{eH}{(2\pi)^2}\int dp\sum_{n\zeta\epsilon}|E|$$

$$\frac{1}{4\sqrt{\pi}}\frac{eH}{(2\pi)^2}\int dp\sum_{n\zeta\epsilon}\int_{1/\Lambda^2}^{+\infty}\frac{ds}{s\sqrt{s}}e^{-sE^2},$$

where Λ is the dimensional cutoff parameter. It should be noted that despite the finiteness of the expression of $\Omega_{\bar{e}} - \Omega_v$, its explicit calculation (as the difference between the divergent objects) also requires the introduction of intermediate regularization to obtain the correct answer in the absence of symmetry between the spectra of particles and antiparticles. In particular, it is possible to introduce the cutoff factor $\theta(\Lambda' - |E|)$, where $\Lambda' > \Lambda$.

3. THE STATE DIAGRAM

To study phase transitions in the system, a numerical investigation of the potential (7) on the minimum relatively independent parameters m and b for different values of the chemical potential μ and the external magnetic field H at $T = 0$ was conducted. The minimum was checked for stability relative to small deviations of b from the direction $\mathbf{H}$. During the calculations, we used the size of the cutoff parameter Λ values, and we denote them by the same symbols as in the original. The maximum value of the relative error was given at 10^{-3}, absolute 10^{-8}.

The dependence of b on the external conditions for $G = 6$ is shown in Fig. 1. The phase diagram for this case is shown in Fig. 2. When $H \longrightarrow 0$ our result continuously transits to the result obtained in [3]. When $H > 0$, a modification of the characteristic structure of the phases[2] for the NJL model in a magnetic field is observed (for details of the latest examples, see, for example, [9]). Spatially inhomogeneous condensate configurations (for which $b \neq 0$) are produced everywhere in the massive phase areas B and C (except for the case where $\mu = 0$), while the parameter b increases gradually with increases of the magnetic field; this also contributes to the further development of a new (discovered in [3]) D phase with strong spatial condensate inhomogeneity at $H \longrightarrow 0$ (in the B and C phases this heterogeneity is absent at $H = 0$). The symmetric massless phase A is limited by the D phase at large values of H.

CONCLUSIONS

Based on the results in this paper, we can discuss the magnetic catalysis of the formation of spatially inhomogeneous configurations of condensates in the framework of the NJL model. It is easy to establish that this phenomenon owes its existence to a feature in the spectrum (6) for $n = 0$, namely, symmetry breaking between the spectra of fermions and antifermions. The removal of terms with $n = 0$ from (8) in numerical calculations leads to the disappearance of the effect. As noted in [3], the linear growth of the parameter b with increasing chemical potential μ is typical for one-dimensional systems; this is consistent with the fact that, as is known, the motion of fermions in a strong external magnetic field has an effectively one-dimensional character.

[2] In this work we did not isolate the oscillation sequences of phases of the same type considered in [9]; we assumed it is at one phase.

ACKNOWLEDGMENTS

The authors thank A. E. Lobanov and A. V. Tukov for their valuable comments and participation in fruitful discussions during our research.

REFERENCES

1. Y. Nambu and G. Jona-Lasinio, Phys. Rev. **124**, 246 (1961).
2. D. V. Deryagin, D. Y. Grigoriev, and V. A. Rubakov, Int. J. Mod. Phys. A **7**, 659 (1992).
3. E. Nakano and T. Tatsumi, Phys. Rev. D: Part. Fields **71**, 114006 (2005).
4. K. Fujikawa, Phys. Rev. D: Part. Fields **21**, 2848 (1980).
5. A. A. Sokolov and I. M. Ternov, *The Relativistic Electron* (Nauka, Moscow, 1983) [in Russian].
6. I. M. Ternov, V. G. Bagrov, and V. Ch. Zhukovskii, Vestn. Mosk. Univ., Fiz. Astron., No. 1, 30 (1966).
7. T. Matsubara, Prog. Theor. Phys. **14**, 351 (1955).
8. J. Schwinger, Phys. Rev. **82**, 664 (1951).
9. D. Ebert, K. G. Klimenko, M. A. Vdovichenko, and A. S. Vshivtsev, Phys. Rev. D: Part. Fields **61**, 025005 (1999).