Quark and pion condensation in a chromomagnetic background field

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The general features of quark and pion condensation in dense quark matter with flavor asymmetry have been considered at finite temperature in the presence of a chromomagnetic background field modelling the gluon condensate. In particular, pion condensation in the case of a constant abelian chromomagnetic field and zero temperature has been studied both analytically and numerically. Under the influence of the chromomagnetic background field the effective potential of the system is found to have a global minimum for a finite pion condensate even for small values of the effective quark coupling constant. In the strong field limit, an effective dimensional reduction has been found to take place.

I. INTRODUCTION

It is well-known that effective field theories with four-fermion interaction (the so-called Nambu – Jona-Lasinio (NJL) models), which incorporate the phenomenon of dynamical chiral symmetry breaking, are quite useful in describing low-energy hadronic processes (see e.g. [1, 2] and references therein). Since the NJL model displays the same symmetries as QCD, it can be successfully used for simulating some of the QCD ground state properties under the influence of external conditions such as temperature, baryonic chemical potential, or even curved space-time etc. [2, 3, 4, 5, 6] In particular, the role of the NJL approach significantly increases, when numerical lattice calculations are not admissible in QCD, like at nonzero baryon density and in the presence of external gauge fields [7, 8]. In this way, it was observed in the framework of a (2+1)-dimensional NJL model that an arbitrary small external magnetic field induces the spontaneous chiral symmetry breaking (CSB) even at arbitrary weak interaction between fermions [9]. Later, it was shown that this phenomenon (the so called magnetic catalysis effect) displays a universal character and can be explained basing on the dimensional reduction mechanism [10, 11] (for a modern status of the magnetic catalysis effect and its applications in different branches of physics, see the reviews [12]).

As an effective theory for low energy QCD, the NJL model does not contain any dynamical gluon fields. However, such a nonperturbative feature of the real QCD vacuum, as the nonzero gluon condensate $\langle F^a_{\mu\nu} F^{a\mu\nu} \rangle$ can be mimicked by external chromomagnetic fields. In particular, for a QCD-motivated NJL model with gluon condensate (i.e. in the presence of an external chromomagnetic field) and finite temperature, it was shown that a weak gluon condensate plays a stabilizing role for the behavior of the constituent quark mass, the quark condensate, meson masses and coupling constants for varying temperature [13]. Then, in a series of papers, devoted to the NJL model with the gluon condensate, it was shown that an external chromomagnetic field, similar to the ordinary magnetic field, serves as a catalyzing factor in the fermion mass generation and dynamical breaking of chiral symmetry as well [14]. The basis for this phenomenon is also the effective reduction of the space dimensionality in the presence of strong external chromomagnetic fields, and this does not depend on the particular form of the field configurations [15].

At present time, it is well-established (see, e.g., [16]) that the gluon condensate is a very slowly decreasing function of the baryon density (baryon chemical potential). So in cold quark matter, it is a nonzero quantity even at sufficiently large baryon densities, which are expected to exist inside the neutron star cores. Evidently, the consideration of the gluon condensate may change significantly

the generally accepted picture of physical processes in the baryon matter. For instance, according to the modern point of view, the color superconducting (CSC) phenomenon can be realized in neutron star cores (see the reviews [4]). However, according to [17], the critical parameters of this phase transition strongly depend on the value of the gluon condensate (in this case, only its chromomagnetic component survives). Moreover, it turns out that even for a rather weak quark coupling, different external chromomagnetic field configurations induce the CSC phenomenon [18]. Quite recently it was found that some combinations of external chromomagnetic and ordinary magnetic fields can penetrate into a bulk of the CSC medium and modify its ground state, producing a new type of color superconductivity [19]. Finally, note that in the dense quark matter, a superstrong magnetic field is originated due to the presence of the gluon condensate [20]. In conclusion, we see that there exist several new physical effects that are intrinsically connected with the gluon condensate (external chromomagnetic fields).

In the present paper, we study the influence of an external chromomagnetic field on the pion condensation phenomenon. This phase transition can also occur in the dense baryon matter, although isotopic asymmetry, with different densities of up and down quarks, is needed for this process to take place. This type of quark matter was already investigated in the framework of NJL type of models, both with and without pion condensation [21, 22, 23, 24, 25]. The main purpose of the present paper is to investigate, in the framework of a NJL model, the behavior of the quark and pion condensation in the presence of an external chromomagnetic field, modelling the gluon condensate. In particular, we will show that, even for a weak coupling of quarks, the pion condensation effect is induced by an external chromomagnetic field and is related to an effective dimensional reduction. The latter effect leads to a nonanalytic logarithmic dependence of the quark and pion condensates on the field strength in the strong field limit.

II. QUARK AND PION CONDENSATES IN EXTERNAL FIELDS

For a system, composed of two flavored quarks, there exists the relation $n_Q = n_{I_3} + n_B/2$ between the electric charge density n_Q, the baryon charge density n_B and the density $n_{I_3} = \frac{1}{2}\bar{q}\tau_3 q$ of the third isospin component $I_3 = \tau_3/2$. Since these quantities are linearly dependent, we will study in the following the isotopic asymmetry (which means that different species of quarks have different densities) in dense quark matter with a nonzero baryon chemical potential μ_B. In this case, the chemical potentials μ_B and μ_I (where μ_I is the isospin chemical potential) are independent parameters. Note that in another possible case, i.e., $\mu_B \neq 0$, $\mu_Q \neq 0$ (the last quantity is the electric charge chemical potential), they are no more independent, but related through the electric charge neutrality condition $n_Q = 0$, if one assumes that quarks are in a weak-equilibrium with electrons, and the whole system is electrically neutral. We restrict ourselves here to the first possibility, i.e. suppose that μ_B and μ_I are independent quantities. In our investigations we also suppose that μ_B is a rather small quantity. In this case, as well as at sufficiently high flavor asymmetry, the color superconductivity effects may be neglected in favour of the normal quark matter or pion superfluidity effects (see the recent discussions in [22, 24]).

A. General definitions

Let us consider a NJL model of flavored and colored quarks $q_{i\alpha}$ ($i = 1, \ldots, N_f$, $\alpha = 1, \ldots, N_c$) with $N_f = 2, N_c = 3$ as numbers of flavors and colors, respectively (for convenience, corresponding indices are sometimes suppressed in what follows), moving in an external chromomagnetic field. The underlying quark Lagrangian is chosen to contain four-quark interaction terms responsible

for spontaneous breaking of chiral and flavor symmetries. Hence, two types of condensates might characterize the ground state of the model: the quark condensate $\langle \bar{q}q \rangle$ (spontaneous breaking of chiral symmetry), and the pion condensate $\langle \bar{q}\gamma_5\tau_1 q \rangle$ (spontaneous breaking of parity and isotopic symmetry). In particular, we consider a Lagrangian which describes dense quark matter with an isotopical asymmetry, and we neglect diquark interactions and hence the possible formation of a diquark condensate. Upon performing the usual bosonization procedure [26], [1] and introducing meson fields σ, $\vec{\pi}$, the four-quark terms are replaced by Yukawa interactions of quarks with these fields, and the Lagrangian takes the following form (our notations refer to four–dimensional Euclidean space with $it = x_4$ [1]):

$$\mathcal{L} = -\bar{q}(i\gamma_\nu \nabla_\nu + i\mu\gamma_0 + \sigma + m + i\gamma^5\vec{\tau}\vec{\pi} + i\mu'\tau_3\gamma^0)q - \frac{1}{4G}(\sigma^2 + \vec{\pi}^2). \tag{1}$$

Here $\mu = (\mu_u + \mu_d)/2$ is the chemical potential averaged over flavors [2], $\mu' = (\mu_u - \mu_d)/2$ is their difference, and G is the (positive) four-quark coupling constant. Furthermore, $\nabla_\mu = \partial_\mu - igA_\mu^a\lambda_a/2$ is the covariant derivative of quark fields in the background field $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{abc}A_\mu^b A_\nu^c$ determined by the potentials A_μ^a ($a = 1, ..., 8$), and $\lambda_a/2$ are the generators of the color $SU_c(3)$ group. Finally, $\vec{\tau} \equiv (\tau^1, \tau^2, \tau^3)$ are Pauli matrices in the flavor space.

Evidently, the Lagrangian (1) with nonvanishing current quark masses, $m \neq 0$, is invariant under the baryon $U_B(1)$ symmetry and the parity transformation P. Moreover, without the μ'-term, it is also invariant under the isospin $SU_I(2)$ group which is reduced to $U_{I_3}(1)$ at $\mu' \neq 0$. At $m = 0$ and $\mu' \neq 0$ the symmetry group of the initial model is $U_B(1) \times U_{I_3L}(1) \times U_{I_3R}(1) \times P$ (here the subscripts L, R mean that the corresponding group acts only on the left, right handed spinors, respectively). It is very convenient to present this symmetry as $U_B(1) \times U_{I_3}(1) \times U_{AI_3}(1) \times P$, where $U_{I_3}(1)$ is the isospin subgroup and $U_{AI_3}(1)$ is the axial isospin subgroup. Quarks are tranformed under these groups in the following ways $q \to \exp(i\alpha\tau_3)q$ and $q \to \exp(i\alpha\gamma^5\tau_3)q$, respectively. In the case of $m = 0$ the phase structure of the model (1) is defined by the competition of only two condensates. One of them is the quark condensate $\langle \bar{q}q \rangle$, and the other is the pion condensate $\langle \bar{q}\gamma_5\tau_1 q \rangle$. If the ground state of the model is characterized by $\langle \bar{q}q \rangle \neq 0$ and $\langle \bar{q}\gamma_5\tau_1 q \rangle = 0$, then the axial symmetry $U_{AI_3}(1)$ of the model is spontaneously broken down, but isospin symmetry $U_{I_3}(1)$ and parity P remain intact. However, if $\langle \bar{q}\gamma_5\tau_1 q \rangle \neq 0$ and $\langle \bar{q}q \rangle = 0$, then only parity P and the isospin symmetry are spontaneously broken down and the pion condensed phase is realized in the model. As a consequence, in the last case we have $\langle \bar{q}\tau_3 q \rangle \neq 0$, i.e., a nonzero difference in densities of up and down quarks arises (isotopic asymmetry of the ground state). Generally speaking, the inverse is not true, i.e., if there is an isotopic asymmetry of quark matter, this does not necessarily mean that pion condensation phenomenon does occur.

In order to investigate the possible generation of quark and pion condensates in the framework of the initial model (1), let us introduce the partition function Z of the system

$$Z = \exp W_E = \int dq d\bar{q} d\sigma d\pi_i \exp\left[\int d^4 x \mathcal{L}\right] \tag{2}$$

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[1] We consider γ−matrices in the 4−dimensional Euclidean space with the metric tensor $g_{\mu\nu} = \text{diag}(-1, -1, -1, -1)$, and the relation between the Euclidean and Minkowski time $x^0_{(E)} = ix^0_{(M)}$: $\gamma^0_{(E)} = i\gamma^0_{(M)}, \gamma^k_{(E)} = \gamma^k_{(M)}$. In what follows we denote the Euclidean Dirac matrices as γ_μ, suppressing the subscript (E). They have the following basic properties $\gamma_\mu^+ = -\gamma_\mu$, $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$. The charge conjugation operation for Dirac spinors is defined as $\psi_c(x) = C(\psi(x)^+)^t$ with $C\gamma_\mu^t C^{-1} = -\gamma_\mu$. We choose the standard representation for the Dirac matrices (see [27]). The γ_5 has the following properties: $\{\gamma^\mu, \gamma_5\} = 0$, $\gamma_5^+ = \gamma_5^t = \gamma_5$. Hence, one finds for the charge-conjugation matrix: $C = \gamma^0\gamma^2$, $C^+ = C^{-1} = C^t = -C$.

[2] It is equal to one third of the baryon chemical potential: $\mu = \mu_B/3$. Since the generator I_3 of the third component of isospin is equal to $\tau_3/2$, the quantity μ' in (1) is half the isospin chemical potential, $\mu' = \mu_I/2$.

with W_E being the Euclidean effective action. Here, the meson fields can be decomposed as follows

$$\sigma = \sigma^{(0)} + \delta\sigma, \;\; \vec{\pi} = \vec{\pi}^{(0)} + \delta\vec{\pi},$$

where $\sigma^{(0)}$ and $\vec{\pi}^{(0)}$ are the condensate fields, which are determined by the minimum of the effective action,

$$\frac{\delta \log Z}{\delta \sigma^{(0)}} = -\langle \bar{q}q + \frac{1}{2G}(\sigma^{(0)} + \delta\sigma)\rangle = 0,$$

$$\frac{\delta \log Z}{\delta \vec{\pi}^{(0)}} = -\langle \bar{q}i\gamma_5\vec{\tau}q + \frac{1}{2G}(\vec{\pi}^{(0)} + \delta\vec{\pi})\rangle = 0.$$

Notice that the above expectation values are expressed through functional integrals over the quark and meson fields, where the latter are calculated by using the saddle point approximation (in what follows, we shall denote these mean values again by σ and $\vec{\pi}$, respectively). Thus, instead of Eq. (2), we shall deal with the following functional integral over fermion fields:

$$Z = \exp W_E = \int dq d\bar{q} \exp\left[\int d^4x \mathcal{L}\right], \tag{3}$$

where now σ and $\vec{\pi}$ in the Lagrangian (1) are understood as constant condensates. In the mean field approximation, where field fluctuations $\delta\sigma$ and $\delta\vec{\pi}$ are neglected, they are given by the following gap equations

$$-\frac{1}{2G}\sigma = \langle\bar{q}q\rangle, \;\; -\frac{1}{2G}\vec{\pi} = \langle i\bar{q}\vec{\tau}\gamma_5 q\rangle. \tag{4}$$

Assume that the only nonvanishing components of the background gauge field potential are for $a = 1, 2, 3$, while others are equal to zero, [3]

$$A_\mu^a \neq 0, \text{ when } a = 1, 2, 3, \text{ and } A_\mu^a = 0, \text{ when } a = 4, \ldots, 8.$$

This implies that only quarks of two colors $\alpha = 1, 2$ do interact with the background field A_μ^a. As a result, the integration over quark degrees of freedom in the partition function (3) is greatly simplified, and we have

$$Z = \text{Det}_{(1)}(i\gamma\partial + \mathcal{M}) \cdot \text{Det}_{(2)}(i\gamma\nabla + \mathcal{M}), \tag{5}$$

where $\mathcal{M} = m + \sigma + i\gamma^5\vec{\pi}\vec{\tau} + i\mu\gamma_0 + i\mu'\tau_3\gamma^0$, and indices (1) and (2) mean that determinants are calculated in the one-dimensional (with color $\alpha = 3$) and in the two-dimensional (with colors $\alpha = 1, 2$) subspaces of the color group, respectively. Assume now that the background field is constant and homogeneous, $F_{\mu\nu}^a = const$. Then the Dirac equations

$$(i\gamma\partial + \mathcal{M})\psi = 0, \qquad (i\gamma\nabla + \mathcal{M})\psi = 0$$

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[3] Of course, if the external constant homogeneous gauge field fills the whole space, then the color as well as the rotational symmetries of the system are broken. This might be considered as an artifact, since in reality there exist both color and rotational invariance (we ignore the color superconductivity effects). In our case, in order to deal with a physically acceptable ground state, i.e. without color symmetry breaking, one can interpret it as a space split into an infinite number of domains with macroscopic extension. Inside each such domain there exist a homogeneous background chromomagnetic field, which generates a nonzero gluon condensate $\langle F_{\mu\nu}^a F^{a\mu\nu}\rangle$. Moreover, the direction of the gauge field varies from domain to domain, so the averaging over all domains results in a zero background chromomagnetic field. Therefore, color as well as rotational symmetries are not broken. Strictly speaking, our following calculations refer to some given macroscopic domain. The obtained results turn out to depend on color and rotational invariant quantities only, and are independent of the particular domain.

have stationary solutions with the energy spectrum ε for quarks of flavor i and color $\alpha = 1, 2, 3$ with quantum numbers k moving in the constant background field $F^a_{\mu\nu}(a = 1, 2, 3)$. In this case, we arrive at the following Euclidean effective action:

$$W_E = \tau \int \frac{dp_4}{2\pi} \sum_{\lambda,\alpha,k,\kappa} \log \left[p_4^2 + (\varepsilon - \kappa\mu)^2 \right] - (\tau L^3) \frac{\sigma^2 + \vec{\pi}^2}{4G}. \tag{6}$$

Here, τ is the imaginary time interval, the sum is over the signs $\lambda = \pm 1$ of the chemical potential μ', the signs $\kappa = \pm 1$ of the chemical potential μ, corresponding to charge conjugate contributions of quarks, color indices $\alpha = 1, 2, 3$, and also over quantum numbers k of quarks, with $\alpha = 3$ for free quarks and the spectrum

$$\varepsilon = \varepsilon_{\vec{p},\lambda} = \sqrt{\left(\sqrt{(\sigma + m)^2 + \vec{p}^2 + \pi_3^2} + \lambda\mu'\right)^2 + \pi_1^2 + \pi_2^2}, \quad (\lambda = \pm 1), \tag{7}$$

and with $\alpha = 1, 2$ for quarks with the spectrum

$$\varepsilon = \varepsilon_{k,\lambda} = \sqrt{\left(\sqrt{(\sigma + m)^2 + \Pi_k^2 + \pi_3^2} + \lambda\mu'\right)^2 + \pi_1^2 + \pi_2^2}, \quad (\lambda = \pm 1), \tag{8}$$

moving in the background color field $F^a_{\mu\nu}$ ($a = 1, 2, 3$). In the above formula, Π_k^2 stands for the eigenvalues of the squared Dirac operator $-(\vec{\gamma}\vec{\nabla})^2$ with $\vec{\nabla} = \vec{\partial} - ig\vec{A}^a\lambda_a/2$.

In the case of finite temperature $T = 1/\beta > 0$, the thermodynamic potential $\Omega = -W_E/(\beta L^3)$ is obtained after substituting $p_4 \to \frac{2\pi}{\beta}(l + \frac{1}{2}), l = 0, \pm 1, \pm 2, ...,$

$$\Omega = -\frac{1}{\beta L^3} \sum_{\lambda,\kappa} \sum_{k,\alpha} \sum_{l=-\infty}^{l=+\infty} \log \left[\left(\frac{2\pi(l + 1/2)}{\beta}\right)^2 + (\varepsilon - \kappa\mu)^2 \right] + \frac{\sigma^2 + \vec{\pi}^2}{4G} + C, \tag{9}$$

where we introduced a subtraction constant C in such a way that at $\sigma = \vec{\pi} = 0$ we have $\Omega = 0$. Next, let us consider the proper time representation

$$\Omega = \frac{1}{\beta L^3} \sum_{\lambda,\kappa} \sum_{k,\alpha} \sum_{l=-\infty}^{l=+\infty} \int_{1/\Lambda_s^2}^{\infty} \frac{ds}{s} \exp\left[-s\left(\frac{2\pi(l + 1/2)}{\beta}\right)^2 - s(\varepsilon - \kappa\mu)^2\right] + \frac{\sigma^2 + \vec{\pi}^2}{4G} + C, \tag{10}$$

where Λ_s is an ultraviolet cutoff ($\Lambda_s \gg \sigma, |\vec{\pi}|$). The temperature dependent contribution can be further transformed with the help of the formula

$$\sum_{l=-\infty}^{+\infty} \exp[-s(2\pi l/\beta + x)^2] = \frac{\beta}{2\sqrt{\pi s}}[1 + 2\sum_{l=1}^{\infty} \exp(-\frac{\beta^2 l^2}{4s}) \cos(x\beta l)], \tag{11}$$

where in our case $x = \frac{\pi}{\beta}$. Then, calculating the quark condensate

$$\langle \bar{q}q \rangle = \frac{\int d\bar{q}dq \ \bar{q}q \exp\left[\int d^4x\mathcal{L}\right]}{\int d\bar{q}dq \exp\left[\int d^4x\mathcal{L}\right]}, \tag{12}$$

and combining the result of (12) and (4), we obtain the following gap equation:

$$\frac{\sigma}{G} = \frac{\sigma + m}{L^3\sqrt{\pi}} \sum_{\lambda,\kappa} \sum_{k,\alpha} \int_{1/\Lambda_s^2}^{\infty} \frac{ds}{\sqrt{s}} \left[1 + 2\sum_{l=1}^{\infty} (-1)^l e^{-\frac{\beta^2 l^2}{4s}}\right] e^{-s(\varepsilon - \kappa\mu)^2} \frac{(\varepsilon - \kappa\mu)\sqrt{\varepsilon^2 - \pi_1^2 - \pi_2^2}}{\varepsilon(\sqrt{\varepsilon^2 - \pi_1^2 - \pi_2^2} - \lambda\mu')}. \tag{13}$$

Here, the first term in the square brackets describes the $T = 0$ contribution (it corresponds to the result of integration over p_4 in the initial equation (6)), while the second term is the finite temperature contribution, $T \neq 0$. The $\vec{\pi}$-condensate can be obtained in a similar way. For π_3 we have

$$\frac{\pi_3}{G} = \frac{\pi_3}{L^3\sqrt{\pi}} \sum_{\lambda,\kappa} \sum_{k,\alpha} \int_{1/\Lambda_s^2}^{\infty} \frac{ds}{\sqrt{s}} \left[1 + 2\sum_{l=1}^{\infty}(-1)^l e^{-\frac{\beta^2 l^2}{4s}} \right] e^{-s(\varepsilon - \kappa\mu)^2} \frac{(\varepsilon - \kappa\mu)\sqrt{\varepsilon^2 - \pi_1^2 - \pi_2^2}}{\varepsilon(\sqrt{\varepsilon^2 - \pi_1^2 - \pi_2^2} - \lambda\mu')}. \quad (14)$$

By comparing this expression with that for σ, we conclude that for nonvanishing quark mass $m \neq 0$ this condensate vanishes, $\pi_3 = 0$. For π_1 (putting $\pi_2 = 0$ by consideration of the symmetry of the problem) we have

$$\frac{\pi_1}{G} = \frac{\pi_1}{L^3\sqrt{\pi}} \sum_{\lambda,\kappa} \sum_{k,\alpha} \int_{1/\Lambda_s^2}^{\infty} \frac{ds}{\sqrt{s}} \left[1 + 2\sum_{l=1}^{\infty}(-1)^l e^{-\frac{\beta^2 l^2}{4s}} \right] \frac{(\varepsilon - \kappa\mu)}{\varepsilon} e^{-s(\varepsilon - \kappa\mu)^2}. \quad (15)$$

It follows from (13) and (15) that at $m = 0$ and $\mu' \neq 0$ there exist only two different solutions of this system of gap equations (except for the trivial one with $\sigma = \pi_1 = 0$), i.e., a) $\sigma = 0, \pi_1 \neq 0$ and b) $\sigma \neq 0, \pi_1 = 0$. Thus we have to find out which of these solutions provide the global minimum of the thermodynamic potential (9) with $\mu' \neq 0$.

In the limit of a vanishing external field ($F_{\mu\nu}^a = 0$), with $\pi_3 = 0$, $\pi_2 = 0$, we have for the particle spectrum

$$\varepsilon_{\vec{p},\lambda} = \sqrt{\left(\sqrt{(\sigma + m)^2 + \vec{p}^2} + \lambda\mu' \right)^2 + \pi_1^2}, \quad (16)$$

and for the sum over quantum states

$$\frac{1}{L^3} \sum_{k,\alpha,\kappa,\lambda} = 12 \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3}$$

for 3 color states ($\alpha = 1, 2, 3$), 2 spin states, and 2 values of $\kappa = \pm 1$. Considering now the case of vanishing temperature $T = 0$, we shall omit the cutoff in the lower limit, $1/\Lambda_s^2 \to 0$, and replace it by the corresponding cutoff in the momentum integration $\vec{p}^2 \leq \Lambda_p^2$. It can be easily seen that the two cutoff parameters Λ_p and Λ_s are related as $\Lambda_s^2 = 2\Lambda_p^2$. Indeed, integration in (6) gives:

$$-\frac{1}{(2\pi)^4} \int dp_4 \int_{\vec{p}^2 < \Lambda_p^2} d^3p \log(\varepsilon^2 + p_4^2) = -\frac{1}{8\pi^2}\Lambda_p^4 + O(\Lambda_p^2)$$

with the momentum cutoff $\vec{p}^2 \leq \Lambda_p^2$, and integration in (10) gives

$$\frac{1}{(2\pi)^4} \int d^4p \int_{1/\Lambda_s^2}^{\infty} \frac{dz}{z} \exp(-iz\varepsilon^2 - izp_4^2) = -\frac{1}{32\pi^2}\Lambda_s^4 + O(\Lambda_s^2)$$

with the proper time cutoff $z \geq \frac{1}{\Lambda_s^2}$. In what follows, we shall denote the Λ_p^2 cutoff by Λ^2, and we shall use it throughout, substituting $\Lambda_s^2 = 2\Lambda^2$. Now, we can perform the proper-time integration in (15) with the help of the formula

$$\sum_{\kappa=\pm 1} \int_0^{\infty} \frac{ds}{\sqrt{s}} \frac{(\varepsilon - \kappa\mu)}{\varepsilon} e^{-s(\varepsilon - \kappa\mu)^2} = \left(\frac{\varepsilon - \mu}{|\varepsilon - \mu|} + 1 \right) \frac{\sqrt{\pi}}{2\varepsilon} = \sqrt{\pi}\frac{\theta(\varepsilon - \mu)}{\varepsilon}. \quad (17)$$

As a result, one obtains for the quark and pion condensates in the vanishing color field background the formulas coinciding with the result of [25],

$$\frac{\sigma}{2G} = 6(\sigma + m) \sum_\lambda \int\limits_{\vec{p}^2 \leq \Lambda^2} \frac{d^3p}{(2\pi)^3} \frac{\theta(\varepsilon - \mu)}{\varepsilon} \frac{\sqrt{(\sigma + m)^2 + \vec{p}^2} + \lambda\mu'}{\sqrt{(\sigma + m)^2 + \vec{p}^2}}, \tag{18}$$

$$\frac{\pi_1}{2G} = 6\pi_1 \sum_\lambda \int\limits_{\vec{p}^2 \leq \Lambda^2} \frac{d^3p}{(2\pi)^3} \frac{\theta(\varepsilon - \mu)}{\varepsilon}. \tag{19}$$

In what follows, we shall analyze the special case of a constant Abelian chromomagnetic field

$$A_\mu^a = \delta_3^a \delta_{\mu 2} x_1 H. \tag{20}$$

The spectrum Π_k^2 of the Dirac operator $-(\vec{\gamma}\vec{\nabla})^2$ has then six branches, two of which correspond to quarks that do not interact with the chromomagnetic field ($\alpha = 3$)

$$\Pi_{1,2}^2 = \vec{p}^2, \tag{21}$$

and the other four correspond to two color degrees of freedom of quarks with "charges" $\pm g/2$ interacting with the external field. The Π_k^2 spectrum of quarks is now given by ($\alpha = 1, 2$)

$$\Pi_{3,4,5,6}^2 = gH\left(n + \frac{1}{2} + \frac{\zeta}{2}\right) + p_3^2, \tag{22}$$

where $\zeta = \pm 1$ is the spin projection on the external field direction, p_3 is the longitudinal component of the quark momentum ($-\infty < p_3 < \infty$),

$$p_\perp^2 = gH\left(n + \frac{1}{2}\right) \tag{23}$$

is the transversal component squared of the quark momentum, and $n = 0, 1, 2, ...$ is the Landau quantum number.

The form of the spectrum is essential for the quark and pion condensate formation. Using the above expressions for energy spectra, we shall next study the quark and pion condensates in the strong field limit.

B. Asymptotic estimates for strong fields

In this section, we consider the special cases of the above configurations of background fields in the strong field limit. Our goal is here to demonstrate that the field is a catalyzing agent for dynamical symmetry breaking, leading to a possible creation of corresponding condensates. The external fields are assumed to be strong as compared to the values of quark $\langle\bar{q}q\rangle$ and pion $\langle\bar{q}\gamma_5\tau_1 q\rangle$ condensates that may be rather small. In this sense, even the expected values of fields simulating the presence of a gluon condensate, which we take to be of the order of $gH = 0.4 - 0.6$ GeV^2, may be considered as strong (these values of the external chromomagnetic field gH correspond to the QCD gluon condensate at zero temperature and zero baryon density [28]). As for the other parameters, we may take their values as in [24, 25], i.e., $\Lambda = 0.65$ GeV, $G = 5.01$ GeV^{-2}.

Consider now the special choice of parameters: $\mu = \pi_2 = \pi_3 = m = 0$. In this simple massless case, though with $\pi_1 \neq 0$, $\sigma \neq 0$, $\mu' \neq 0$, the thermodynamic potential (10) takes the form

$$\Omega = \frac{1}{2\sqrt{\pi}L^3} \sum_{\lambda,\kappa} \sum_{k,\alpha} \int\limits_{\frac{1}{2\Lambda^2}}^{\infty} \frac{ds}{s^{\frac{3}{2}}} \left[1 + 2\sum_{l=1}^{\infty}(-1)^l e^{-\frac{\beta^2 l^2}{4s}}\right] e^{-s\varepsilon^2} + \frac{\sigma^2 + \pi_1^2}{4G} + C. \tag{24}$$

Next, consider the zero temperature case, $T = 0$. Then the second term in the square bracket in the above expression vanishes and we are left with the first term equal to unity. The above equation describes the effective potential $V_{\text{eff}} = \Omega|_{T=0}$

$$V_{\text{eff}}(\sigma, \pi_1) = \frac{1}{2\sqrt{\pi}} \int\limits_{\frac{1}{2\Lambda^2}}^{\infty} \frac{ds}{s^{\frac{3}{2}}} \left[4 \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \, e^{-s\varepsilon_{\vec{p},\lambda}^2} + \frac{1}{L^3} \sum_{\kappa} \sum_{k,\lambda,\alpha=1,2} e^{-s\varepsilon_{k,\lambda}^2} \right] + \frac{\sigma^2 + \pi_1^2}{4G} + C. \quad (25)$$

The first term in the square bracket stands for quarks with $\alpha = 3$, when we have

$$\varepsilon_{\vec{p},\lambda}^2 = (\sqrt{\sigma^2 + \vec{p}^2} + \lambda\mu')^2 + \pi_1^2, \quad (26)$$

and 4 stands for two values of $\kappa = \pm 1$ and two values of spin projection $\zeta = \pm 1$. The second term stands for quarks with $\alpha = 1, 2$, where we have

$$\varepsilon_{k,\lambda}^2 = (\sqrt{\sigma^2 + \Pi_k^2} + \lambda\mu')^2 + \pi_1^2. \quad (27)$$

Let us first consider the symmetric case, when the chemical potentials for the u-quark and the d-quark are equal, $\mu' = (\mu_u - \mu_d)/2 = 0$. Then, the energy of the free quark with $\alpha = 3$ becomes

$$\varepsilon_{\vec{p},\lambda}^2 = \vec{p}^2 + \sigma^2 + \pi_1^2 = \vec{p}^2 + \sigma_1^2,$$

where $\sigma_1^2 = \sigma^2 + \pi_1^2$.

Consider now the case, when the Abelian-like background field is strong enough, $gH = O(\Lambda^2)$, but $gH < 2\Lambda^2$. The momentum squared Π_k^2 is given by (22), and summation over quark quantum numbers in the chromomagnetic field gives

$$\frac{1}{L^3} \sum_{\lambda} \sum_{k,\alpha} = 2\frac{gH}{4\pi} \sum_{\lambda} \sum_{n=0}^{\infty} (2 - \delta_{n0}) \int \frac{dp_3}{2\pi}. \quad (28)$$

Now, the momentum integral in the first term in (25) gives

$$\int \frac{d^3p}{(2\pi)^3} \, e^{-s\varepsilon_{\vec{p},\lambda}^2} = \frac{\sqrt{\pi}}{8\pi^2} \frac{e^{-s\sigma_1^2}}{s^{3/2}}.$$

In the second term in (25) for quarks with $\alpha = 1, 2$, due to the strong field condition, we take only the contribution of the term with $n = 0$ in the sum over Landau quantum numbers, while the contribution of large quantum numbers is described similar to the contribution of the free quark term, and thus we must take the free term three times. As a result, after integration in the second term over the p_3-component of the quark momentum we have

$$V_{\text{eff}}(\sigma_1) = \frac{1}{4\pi^2} \left[\int\limits_{\frac{1}{2\Lambda^2}}^{\infty} ds \, e^{-s\sigma_1^2} \frac{3}{s^3} + \int\limits_{\frac{1}{gH}}^{\infty} ds \, e^{-s\sigma_1^2} \frac{gH}{2s^2} \right] + \frac{\sigma_1^2}{4G} + C.$$

The cutoff momentum Λ as well as the background field gH are large $\Lambda \gg \sigma_1$, $gH \gg \sigma_1^2$ and, hence we can approximately integrate over s in the first and the second terms according to the following formulas:

$$\int\limits_{x_{\min}}^{\infty} \frac{dx}{x^3} e^{-x} = \frac{1}{2x_{\min}^2} - \frac{1}{x_{\min}} - \frac{1}{2} \log x_{\min} + \text{const}, \quad \text{where } x_{\min} = \frac{\sigma_1^2}{2\Lambda^2} \ll 1, \quad (29)$$

$$\int\limits_{x_{\min}}^{\infty} \frac{dx}{x^2}\, e^{-x} = \frac{1}{x_{\min}} + \log x_{\min} + \text{const}, \quad \text{where } x_{\min} = \frac{\sigma_1^2}{gH} \ll 1. \tag{30}$$

As a result we find the effective potential

$$V_{\text{eff}}(\sigma_1) = \frac{1}{4\pi^2}\left\{ 3\sigma_1^4 \left[\frac{1}{2}\left(\frac{2\Lambda^2}{\sigma_1^2}\right)^2 - \frac{2\Lambda^2}{\sigma_1^2} + \frac{1}{2}\log\frac{2\Lambda^2}{\sigma_1^2} + C_1 \right] + \frac{gH\sigma_1^2}{2}\left(\frac{gH}{\sigma_1^2} + \log\frac{\sigma_1^2}{gH} + C_2 \right) \right\} + \frac{\sigma_1^2}{4G} + C, \tag{31}$$

where C_1, C_2 are certain numerical constants. Now, let us find the minimum σ_0 of the thermodynamic potential (31). Then, the minimum σ_0 is described by the solution of the equation, where we neglect the terms that do not contain large parameters Λ^2 or $gH\log(gH/\sigma_0^2)$,

$$\frac{\partial V_{\text{eff}}}{\partial \sigma_1^2}\Big|_{\sigma_1 = \sigma_0} = \frac{1}{4\pi^2}\left[-6\Lambda^2 - 3\sigma_0^2 \log\frac{\sigma_0^2}{2\Lambda^2} + \frac{gH}{2}\log\frac{\sigma_0^2}{gH} \right] + \frac{1}{4G} = 0.$$

Within the approximation of strong background field

$$gH\log(gH/\sigma_0^2) \gg \sigma_0^2 \log(2\Lambda^2/\sigma_0^2), \tag{32}$$

the above equation simplifies to the following form:

$$1 = \frac{6G\Lambda^2}{\pi^2} + G\frac{gH}{2\pi^2}\log\frac{gH}{\sigma_0^2} \tag{33}$$

with the solution

$$\sigma_0 = \sqrt{\frac{gH}{2\pi}}\, \exp\left(-\frac{2\pi^2(1 - \tilde{g})}{GgH} \right), \tag{34}$$

where the effective coupling constant,

$$\tilde{g} = \frac{6G\Lambda^2}{\pi^2}, \tag{35}$$

can be arbitrary small $\tilde{g} < 1$, contrary to the zero magnetic field case $H = 0$, when it should be $\tilde{g} > 1$ in order that the nontrivial solution for σ_0 exists. This is a very interesting result. Indeed, we see that at $\mu' = 0$ and arbitrary small attraction between quarks ($\tilde{g} < 1$) the external chromomagnetic field catalyzes the dynamical chiral symmetry breaking and quarks acquire a nonzero mass. which is equal to σ_0 (34). In contrast, if $H = 0$ and $\tilde{g} < 1$, the quarks remain massless and symmetry is unbroken. After removing the cutoff with the use of (33), the effective potential (31) takes the form

$$V_{\text{eff}}(\sigma_1) = \frac{gH\sigma_1^2}{8\pi^2}\left(\log\frac{\sigma_1^2}{\sigma_0^2} - 1 \right), \tag{36}$$

which is just the effective potential of the two-dimensional Gross-Neveu model (see, e.g., [29]). This result shows that the chromomagnetic catalysis effect is accompanied by the effective reduction of the space-time dimensionality.

Next, consider the case of flavor asymmetric quark matter with $\mu' \neq 0$. For simplicity, μ is taken to be zero as before. In this case (recall that $m = 0$), there are two nontrivial solutions of the gap equations (13) and (15) (see the remark after (15)), which are the points on the σ- or π_1-axes. So, in order to find the global minimum point, it is sufficient (and very convenient) to reduce the

investigation of the effective potential $V_{\text{eff}}(\sigma, \pi_1)$ (25) as a function of two variables σ and π_1 to two particular cases. First, we shall study V_{eff} as a function of π_1 only with $\sigma = 0$, $\pi_1 \neq 0$, then as a function of σ with $\sigma \neq 0$, $\pi_1 = 0$. When comparing these two particular cases, one can obtain the genuine global minimum point of the effective potential $V_{\text{eff}}(\sigma, \pi_1)$. To simplify our calculation, we assume that μ' and gH are large, $\mu'^2 = O(\Lambda^2)$ and $gH = O(\Lambda^2)$.

a) The case $\sigma = 0$, $\pi_1 \neq 0$.

In this case the first term in (25),

$$V_{\text{eff}}^{(1)}(\pi_1) = \frac{1}{2\sqrt{\pi}} \int\limits_{\frac{1}{2\Lambda^2}}^{\infty} \frac{ds}{s^{\frac{3}{2}}} \left[4 \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \, e^{-s\varepsilon_{\vec{p},\lambda}^2} \right] \tag{37}$$

can be expressed through the following approximation of the integral (with $s\mu'^2 \gg 1$):

$$\sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \, e^{-s\varepsilon_{\vec{p},\lambda}^2} = \frac{\sqrt{\pi}\, e^{-s\pi_1^2}}{4\pi^2} \left(2\mu'^2 + \frac{1}{2s} \right) \frac{1}{\sqrt{s}}. \tag{38}$$

Integration over s should be approximately performed separately for the first and second terms in (38) with different lower limits

$$V_{\text{eff}}^{(1)}(\pi_1) = \frac{1}{4\pi^2} \left(4 \int\limits_{\frac{1}{\mu'^2}}^{\infty} \frac{ds\,\mu'^2}{s^2} \, e^{ts - s\pi_1^2} + 3 \int\limits_{\frac{1}{2\Lambda^2}}^{\infty} \frac{ds}{s^3} \, e^{-s\pi_1^2} \right).$$

Here, we also added the contribution of quarks with $\alpha = 1, 2$ with high Landau quantum numbers $n \gg 1$.

In the second term in (25),

$$V_{\text{eff}}^{(2)}(\pi_1) = \frac{1}{2\sqrt{\pi}} \int\limits_{1/\Lambda_s^2}^{\infty} \frac{ds}{s^{\frac{3}{2}}} \frac{1}{L^3} \sum_{\kappa} \sum_{k,\lambda,\alpha=1,2} e^{-s\varepsilon_{k,\lambda}^2},$$

with $gH = O(\Lambda^2)$ and $\mu' = O(\Lambda)$, but $gH < \mu'^2 < 2\Lambda^2$, we should estimate the contribution of the two main terms, $n = 0$ and $n = 1$, in the sum over n. The $n = 0$ contribution is expressed through the integral

$$\sum_{\lambda} \int \frac{dp_3}{(2\pi)} \exp[-s(\pi_1^2 + (|p_3| + \lambda\mu')^2)] = \frac{1}{\pi} \, e^{-s\pi_1^2} \sqrt{\frac{\pi}{s}}, \tag{39}$$

and summation over the remaining quantum numbers is made according to (28) with $n = 0$. For the $n = 1$ term, we have a new expression with integration of the following exponential term:

$$\sum_{\lambda} \int \frac{dp_3}{(2\pi)} \exp[-s(\pi_1^2 + (\sqrt{gH + p_3^2} + \lambda\mu')^2)]. \tag{40}$$

In order to estimate the $n = 1$ contribution, we write $p_3 = \bar{p}_3 + \psi$, where ψ is a small deviation of p_3 from the minimum of the exponential quantity $\sqrt{\bar{p}_3^2 + gH} = \mu'$ for $\lambda = -1$. Then the integral in (40) for $n = 1$ is estimated as

$$\sum_{\lambda} \int \frac{d\psi}{2\pi} \exp[-s\frac{\mu'^2 - gH}{\mu'^2}\psi^2] = \frac{1}{\pi} \sqrt{\frac{\pi}{s}} \frac{\mu'}{\sqrt{\mu'^2 - gH}}. \tag{41}$$

Recall that this estimation is justified under the above assumption, i.e., $gH = O(\Lambda^2)$ and $\mu' = O(\Lambda)$, but $gH < \mu'^2 < 2\Lambda^2$. Finally, summing up the $n = 0$ and $n = 1$ contributions (39) and (41), we find

$$V_{\text{eff}}^{(2)}(\pi_1) = \frac{gH}{8\pi^2}\left(1 + \frac{2\mu'}{\sqrt{\mu'^2 - gH}}\right)\int\limits_{\frac{1}{gH}}^{\infty}\frac{ds}{s^2}\,e^{-s\pi_1^2}.$$

In the second term in the above bracket, we took into account that the state with $n = 1$ is two-fold degenerate in the spin variable. Summing up all the contributions, and taking into account the integrals (29) and (30), we have

$$V_{\text{eff}}(\pi_1) = \frac{1}{4\pi^2}\left\{\frac{3\pi_1^4}{2}\left[(\frac{2\Lambda^2}{\pi_1^2})^2 - 2\frac{2\Lambda^2}{\pi_1^2} - \log\frac{\pi_1^2}{2\Lambda^2} + C_1\right] + \left[4\mu'^4 + \frac{(gH)^2}{2}\left(1 + \frac{\mu'}{\sqrt{\mu'^2 - gH}}\right)\right] + \right.$$

$$\left. + 4\mu'^2\pi_1^2\left(\log\frac{\pi_1^2}{\mu'^2} + C_2\right) + \frac{gH\pi_1^2}{2}\left(1 + \frac{\mu'}{\sqrt{\mu'^2 - gH}}\right)\left(\log\frac{\pi_1^2}{gH} + C_3\right)\right\} + \frac{\pi_1^2}{4G} + C. \quad (42)$$

The minimum point π_0 of this function obeys the following stationarity equation $\partial V_{\text{eff}}/\partial\pi_1^2|_{\pi_1=\pi_0} = 0$, which gives

$$1 = \frac{6G\Lambda^2}{\pi^2} - \frac{4\mu'^2 G}{\pi^2}\log\frac{\pi_0^2}{\mu'^2} - \frac{gH}{2\pi^2}G\left(1 + \frac{\mu'}{\sqrt{\mu'^2 - gH}}\right)\log\frac{\pi_0^2}{gH}, \quad (43)$$

where the approximation of strong background field, $gH\log(gH/\pi_1^2) \gg \pi_1^2\log(2\Lambda^2/\pi_1^2)$, has been used, similar to (32). After substituting the above equation into (42), the effective potential takes the form

$$V_{\text{eff}}(\pi_1^2) = \frac{\pi_1^2}{4\pi^2}\left[4\mu'^2 + \frac{gH}{2}\left(1 + \frac{\mu'}{\sqrt{\mu'^2 - gH}}\right)\right]\left(\log\frac{\pi_1^2}{\pi_0^2} - 1\right), \quad (44)$$

which again, like (36), resembles the effective potential in the two-dimensional Gross – Neveu model [29]. Note also that the minimum π_0 of the function (42), under our assumption of a strong color background field, exists even for weak coupling of quarks, i.e., at $\tilde{g} < 1$, while with zero background field, it can exist only if $\tilde{g} > 1$. Next, let us study the behaviour of the effective potential (25) as a function of σ, when $\pi_1 = 0$.

b) The case $\sigma \neq 0, \pi_1 = 0$.

Here again we use the above assumption $gH = O(\Lambda^2)$ and $\mu' = O(\Lambda)$, but $gH < \mu'^2 < 2\Lambda^2$. The term in the effective potential with $\alpha = 3$ is determined by the expression

$$V_{\text{eff}}^{(1)} = \frac{1}{2\sqrt{\pi}}\int\limits_{1/\Lambda_s^2}^{\infty}\frac{ds}{s^{\frac{3}{2}}}\left[4\sum_\lambda\int\frac{d^3p}{(2\pi)^3}\,e^{-s\varepsilon_{\vec{p},\lambda}^2}\right], \quad (45)$$

where $\varepsilon_{\vec{p},\lambda} = \sqrt{\sigma^2 + \vec{p}^2} + \lambda\mu'$. The main contribution to the integral over p, for large μ', is given by large p near $\bar{p} = \sqrt{\mu'^2 - \sigma^2}$. Then the lower limit in the integral over s should be taken as $\frac{1}{\mu'^2 - \sigma^2}$, and we obtain, with $p = \bar{p} + \psi$,

$$V_{\text{eff}}^{(1)} = \frac{\bar{p}^2}{\pi^{5/2}}\int\limits_{\frac{1}{\mu'^2 - \sigma^2}}^{\infty}\frac{ds}{s^{3/2}}\int d\psi\,e^{-s(\psi\bar{p}/\mu')^2} = \frac{\bar{p}\mu'}{\pi^2}\int\limits_{\frac{1}{\mu'^2 - \sigma^2}}^{\infty}\frac{ds}{s^2} = \frac{1}{\pi^2}\mu'(\mu'^2 - \sigma^2)^{\frac{3}{2}}. \quad (46)$$

Then, the first term, where the contribution of quarks with $\alpha = 1, 2, 3$ in the region of very large values of $s \gg \frac{1}{\mu'^2}, \frac{1}{gH}$ is also added, is estimated as

$$V_{\text{eff1}}(\sigma) = \frac{1}{\pi^2}(\mu'^2 - \sigma^2)^{\frac{3}{2}}\mu' + \frac{3\sigma^4}{8\pi^2}\left[\left(\frac{2\Lambda^2}{\sigma^2}\right)^2 - 4\frac{\Lambda^2}{\sigma^2} - \log\frac{\sigma^2}{2\Lambda^2} + C_1\right]. \tag{47}$$

Consider now the contribution of the term with $\alpha = 1, 2$ in the region of comparatively small values of s close to the threshold $1/gH$,

$$V_{\text{eff2}} = \frac{1}{4\sqrt{\pi}}\int\limits_{1/gH}^{\infty}\frac{ds}{s^{\frac{3}{2}}}\sum_{\lambda}\frac{2gH}{4\pi}\sum_{n}(2 - \delta_{n0})\int\frac{dp_3}{2\pi}\exp[-s(\sqrt{\sigma^2 + p_3^2 + gHn} + \lambda\mu')^2]. \tag{48}$$

Since we assume that μ'^2 and gH are of the same order of magnitude and large, $\mu'^2 = O(\Lambda^2)$ $gH = O(\Lambda^2)$, two terms, $n = 0$ and $n = 1$, will make the main contribution to the sum over n. The $n = 0$ term is determined by the integral, approximately equal to

$$\int\frac{dp_3}{2\pi}\exp[-s(\sqrt{\sigma^2 + p_3^2} - \mu')^2] \approx \frac{\mu'}{2\sqrt{\pi s}\sqrt{\mu'^2 - \sigma^2}}.$$

The $n = 1$ term gives

$$2\sum_{\lambda}\int\frac{dp_3}{2\pi}\exp[-s(\sqrt{\sigma^2 + p_3^2 + gH} + \lambda\mu')^2] \approx \frac{\mu'}{\sqrt{\pi s}\sqrt{\mu'^2 - \sigma^2 - gH}},$$

where we had introduced a factor two due to the two-fold degeneracy of the $n = 1$ Landau level in the spin variable. The above two terms with $n = 0$ and $n = 1$ should be summed up to give

$$V_{\text{eff2}}(\sigma) = \frac{gH\mu'}{8\pi^2}\left(\frac{1}{\sqrt{\mu'^2 - \sigma^2}}\int\limits_{\frac{1}{\mu'^2 - \sigma^2}}^{\infty}\frac{ds}{s^2} + \frac{2}{\sqrt{\mu'^2 - gH - \sigma^2}}\int\limits_{\frac{1}{\mu'^2 - \sigma^2 - gH}}^{\infty}\frac{ds}{s^2}\right)$$

$$= \frac{gH\mu'}{8\pi^2}\left(\sqrt{\mu'^2 - \sigma^2} + 2\sqrt{\mu'^2 - gH - \sigma^2}\right). \tag{49}$$

The effective potential after summing the contributions V_{eff1} and V_{eff2} is finally estimated as follows:

$$V_{\text{eff}}(\sigma) = \frac{\mu'}{\pi^2}(\mu'^2 - \sigma^2)^{\frac{3}{2}} + \frac{3\sigma^4}{8\pi^2}\left[\left(\frac{2\Lambda^2}{\sigma^2}\right)^2 - 4\frac{\Lambda^2}{\sigma^2} - \log\frac{\sigma^2}{2\Lambda^2} + C_1\right]$$

$$+ \frac{gH\mu'}{8\pi^2}\left(\sqrt{\mu'^2 - \sigma^2} + 2\sqrt{\mu'^2 - gH - \sigma^2}\right) + \frac{\sigma^2}{4G} + C. \tag{50}$$

Next, we have to find the minimum point σ_0 of this effective potential, i.e. to solve the stationarity equation $\partial V_{\text{eff}}/\partial\sigma^2 = 0$, which gives for σ_0 the equation

$$-\frac{3\mu'^2}{2\pi^2} - \frac{3\Lambda^2}{2\pi^2} - \frac{3\sigma_0^2}{4\pi^2}\log\frac{\sigma_0^2}{2\Lambda^2} - \frac{gH}{16\pi^2}\left(1 + \frac{2\mu'}{\sqrt{\mu'^2 - gH}}\right) + \frac{1}{4G} + O(\sigma^2) = 0.$$

It is seen that, contrary to the case with the pion condensate, there are no large terms like $gH \log(gH/\sigma^2)$ in the above equation, and hence, for large terms with Λ_s, it has a nontrivial solution $\sigma_0 \neq 0$ only if $\tilde{g} > 1$, which is the same condition as for the zero field case (see, e.g., [24]),

$$\sigma_0 = \sqrt{\tilde{C}}\Lambda \exp\left(-\frac{\Lambda^2}{\sigma_0^2}(1 - \frac{1}{\tilde{g}})\right), \tag{51}$$

where $\tilde{C}$ is a certain numerical constant of order unity.

Comparing the results of the particular considerations **a)** and **b)**, the following preliminary conclusions can be made. It is clear that at very small values of the coupling constant G, i.e. at $\tilde{g} < 1$, and for $\mu' > 0$, both the isotopic and the chiral symmetry of the model (1) are not broken down at $H = 0$, and quarks are massless particles (recall, we consider only the case with zero current quark mass). However, if an external chromomagnetic field H is present (and it is rather strong in our analytical consideration), then the effective potential (25) of the system acquires a nontrivial global minimum point which lies on the π_1-axis and has the form $(\sigma = 0, \pi_1 = \pi_0 \neq 0)$, where π_0 is the solution of the stationarity equation (43). This means that in this case a nonzero pion condensate $\langle \bar{q}\gamma_5\tau_1 q \rangle = -\pi_0/2G$ is generated by an external chromomagnetic field (chromomagnetic catalysis of the pion condensation), and the isotopic symmetry $U_{I_3}(1)$ of the model is spontaneously broken down (see the text after (1)). Moreover, since the expression for the effective potential in this case resembles the effective potential in the 2-dimensional Gross – Neveu model, one can say that this effect, similar to the magnetic catalysis phenomenon, is provided by the dimensional reduction mechanism.

Let us present some numerical calculations in the next section, in order to support this conclusion.

III. NUMERICAL RESULTS

Consider first the case of a zero external field $F_{\mu\nu} = 0$, and flavor symmetric medium $\mu' = 0$. Moreover, the temperature and bare quark mass m are taken to be zero throughout the present section. The numerical analysis of the behavior of the effective potential (25), which is just the thermodynamic potential Ω for $T = 0$, was performed for the value of the coupling constant $G = 5.01$ GeV^{-2}, taken from [24, 25], and for the quark chemical potential $\mu = 0$ (see Fig. 1). In this case the quark energy spectrum is symmetric with respect to σ and π_1, i.e. the function $V_{\text{eff}}(\sigma, \pi_1)$ depends only on the single variable $\sqrt{\sigma^2 + \pi_1^2}$. Hence we draw the effective potential as a function of only one variable, e.g., σ. In this figure, the picture on the left corresponds to the value of the cutoff parameter $\Lambda = 0.165$ GeV (in this case $\tilde{g} = 0.08$), whereas the right picture corresponds to $\Lambda = 0.65$ GeV ($\tilde{g} = 1.29$). The figure demonstrates the general feature of the effective potential at $\mu' = 0$ and $gH = 0$. Namely, if the effective coupling constant $\tilde{g}$ is sufficiently small, $\tilde{g} < 1$, then the global minimum point of the effective potential is at the origin, and the chiral symmetry is not broken (left picture). However, if $\tilde{g} > 1$, the effective potential has a nontrivial global minimum, and the chiral symmetry is spontaneously broken down (right picture). The dependence of $V_{\text{eff}}(\sigma, \pi_1)$ on the two variables σ and π_1 ($\tilde{g} = 1.29$) is depicted in Fig. 2 for $\mu = 0$. The picture is evidently symmetric, as it should be in this flavor symmetric case.

It is also necessary to note that if $\tilde{g} > 1$, then at $gH = 0$ and $\mu' > 0$ a nonzero pion condensate appears in the model (1) [24, 25]. (In this case, the isotopic symmetry $U_{I_3}(1)$ is broken down). However, if $\tilde{g} < 1$, then at $gH = 0$ and arbitrary values of μ', including zero, the symmetry of the NJL model (1) remains intact.

Now, in order to present numerical arguments in favour of the chromomagnetic catalysis of pion condensation, we will study the effective potential $V_{\text{eff}}(\sigma, \pi_1)$ (25) at $\mu' > 0$ and suppose that

$gH = 0.5$ GeV^2 (this value of gH mimicks the nonzero QCD gluon condensate [28]). Since the current quark mass m is equal to zero, the nontrivial stationary points of the function (25) lie either on the σ or on π_1 axis (see the remark following formula (15)). Thus, in order to find the global minimum point of the effective potential (25), it is enough to consider the behavior of $V_{\text{eff}}(\sigma, \pi_1)$ along the coordinate σ and π_1-axis only. The corresponding curves are presented in Fig. 3 for $\mu' = 0.15$ GeV, and physical values of $G = 5.01$ GeV^{-2} and $\Lambda = 0.65$ GeV (in this case $\tilde{g} = 1.29$). They demonstrate that, although a nontrivial stationary point of the thermodynamic potential $\Omega = V_{\text{eff}}$ at $T = 0$ does exist on the σ-axis in a chromomagnetic field gH (and this is just a saddle point in this case), the value of the effective potential at the other stationary point, $\sigma = 0, \pi_1 = \pi_0 \neq 0$, is slightly deeper. Our calculations indicate that with growing μ' the difference between the minima of the curves for $V_{\text{eff}}(\pi_1)$ and $V_{\text{eff}}(\sigma)$ increases, and the very minima get deeper. The comparison of the results obtained with the known results at zero background field, demonstrated that the background field deepens the minimum for the pion condensate. Moreover, we formally considered the interesting case of a smaller value of Λ, corresponding to $\tilde{g} < 1$. The calculated effective potential then has a minimum at a nontrivial value of π_0 for weak coupling of quarks for $\tilde{g} < 1$, as well. The corresponding curves are presented in Figs. 4, 5 for different values of $\mu' = 0.1$ GeV, $\mu' = 0.15$ GeV, respectively, and $G = 5.01$ GeV^{-2} and $\Lambda = 0.46$ GeV (in this case $\tilde{g} = 0.64$). [4] Thus, only at this point, $\sigma = 0, \pi_1 = \pi_0 \neq 0$, the global minimum of the thermodynamic potential (25) is observed, and the nonzero pion condensate $\langle \bar{q} \gamma_5 \tau_1 q \rangle = -\pi_0/2G$ is generated by the external chromomagnetic field in the model.
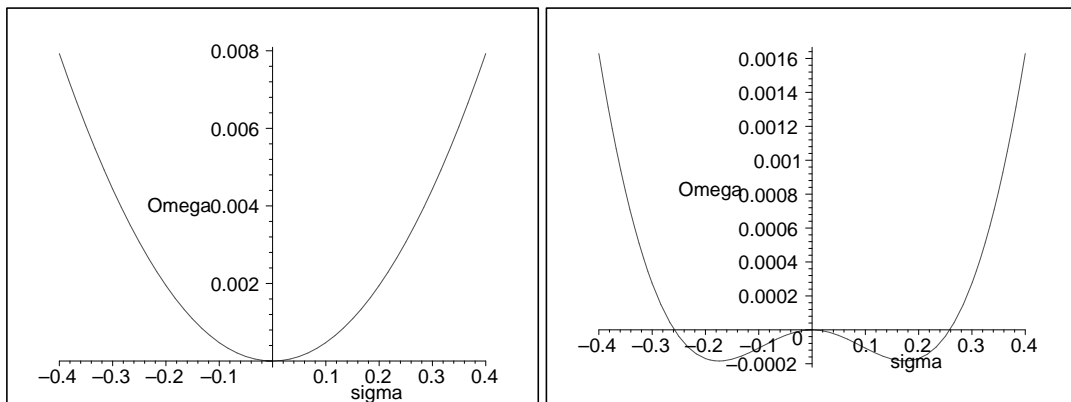

FIG. 1: Thermodynamic potential $\Omega = V_{\text{eff}}(\sigma, 0)$ at $T = 0$, $\mu' = 0$, $\mu = 0$ and $G = 5.01$ GeV^{-2} as a function of σ (in GeV-units) for $\Lambda = 0.165$ GeV – left picture, and $\Lambda = 0.65$ GeV – right picture.

IV. CONCLUSIONS

We considered, in the framework of an NJL model, the effects of quark and pion condensation in dense quark matter with and without flavor asymmetry under the influence of an external chromomagnetic field modelling the gluon condensate $\langle F^a_{\mu\nu} F^{a\mu\nu} \rangle$. The general conclusion is that the presence of the chromomagnetic field catalyses the effect of quark or pion condensation, depending on the values of the chemical potential μ'. Indeed, when there is no chromomagnetic field, no

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[4] In contrast with the numerical results, in our analytical investigations (see the previous section), the stationary point of the effective potential was not observed on the σ-axis at $\tilde{g} < 1$. This fact can be explained due to a different domain for the variable μ' used in these cases. Indeed, it was supposed in section II.B that $\mu' = O(\Lambda)$, whereas for the numerical consideration, we used $\mu' \ll \Lambda$.
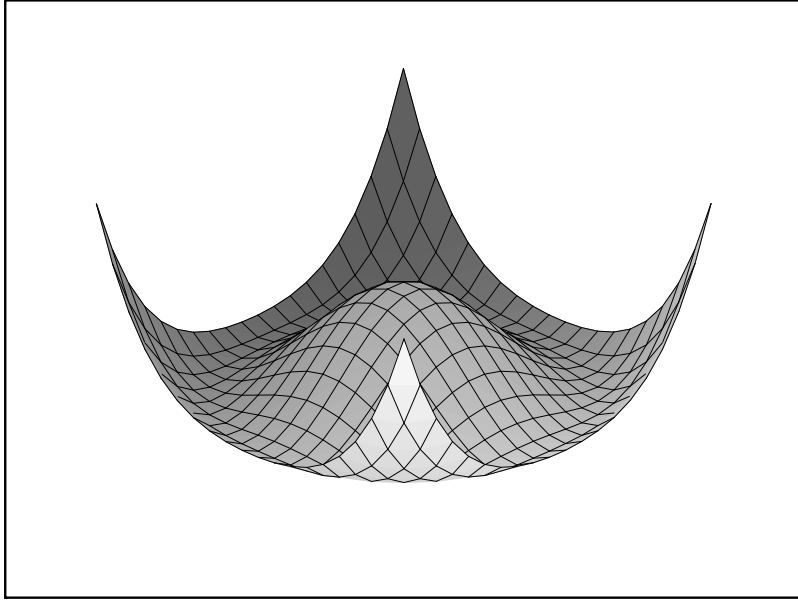
FIG. 2: Thermodynamic potential $\Omega = V_{\text{eff}}(\sigma, \pi_1)$ at $T = 0$, $gH = 0$, $\mu' = 0$ as a function of σ, π_1 for $\Lambda_p = 0.65, \mu = 0$.
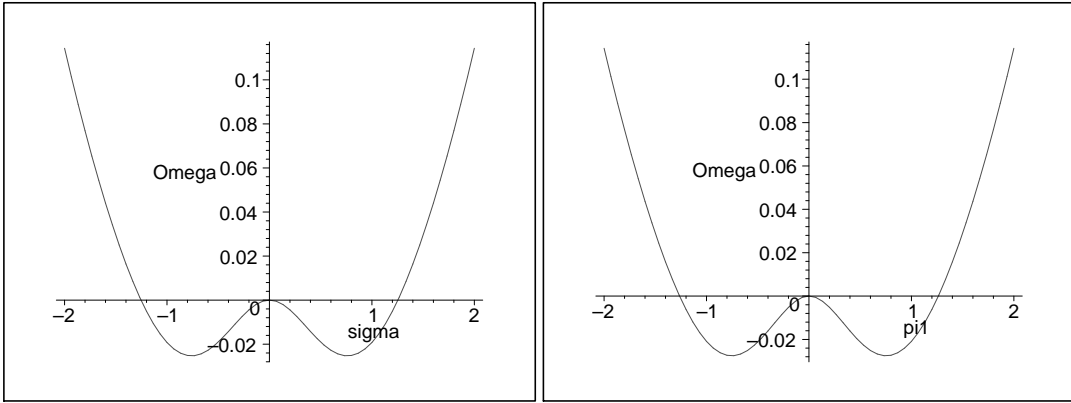

FIG. 3: Thermodynamic potential $\Omega = V_{\text{eff}}(\sigma, \pi_1)$ at $T = 0, \mu = 0$, $gH = 0.5$ GeV^2, $\Lambda = 0.65$ GeV, $\tilde{g} = 1.29$, $\mu' = 0.15$ GeV as a function of σ (in GeV-units) at $\pi_1 = 0$ – left picture, and of π_1 (in GeV) at $\sigma = 0$ – right picture.
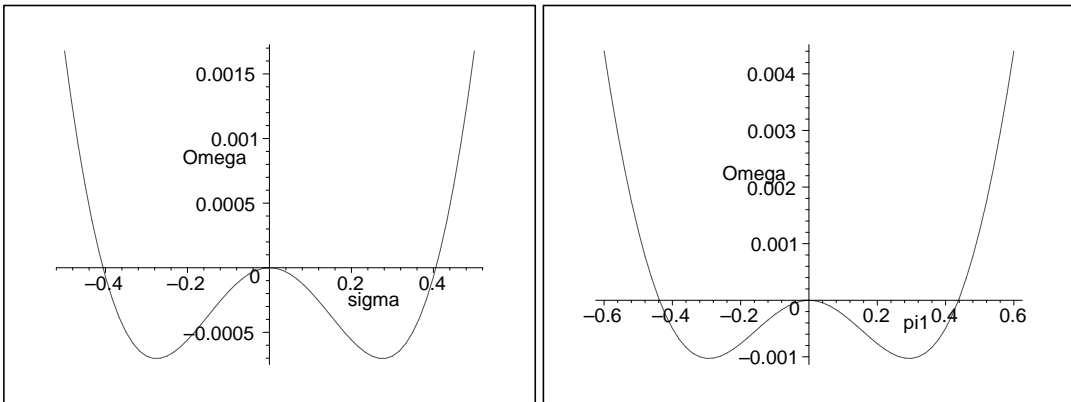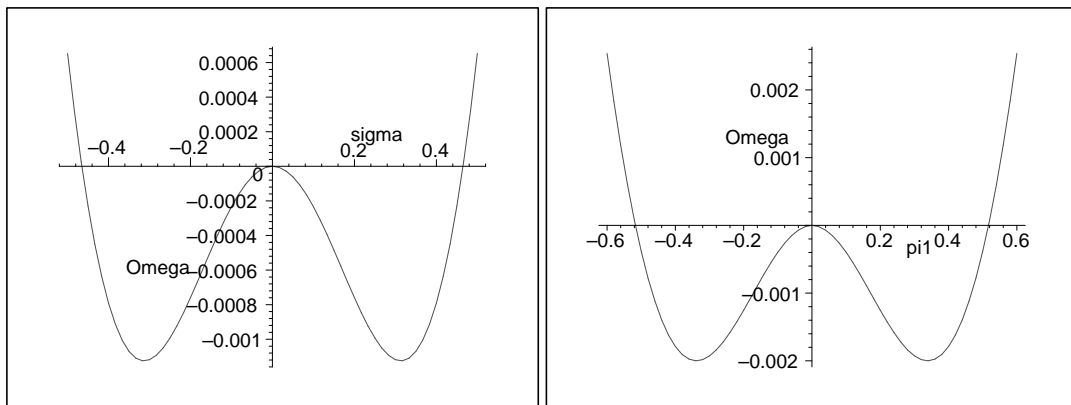

FIG. 4: Thermodynamic potential $\Omega = V_{\text{eff}}(\sigma, \pi_1)$ at $T = 0, \mu = 0$, $gH = 0.5$ GeV^2, $\Lambda = 0.46$ GeV, $\tilde{g} = 0.64$, $\mu' = 0.1$ GeV as a function of σ (in GeV-units) at $\pi_1 = 0$ – left picture, and of π_1 (in GeV) at $\sigma = 0$ – right picture.

FIG. 5: Thermodynamic potential $\Omega = V_{\text{eff}}(\sigma, \pi_1)$ at $T = 0, \mu = 0$, $gH = 0.5$ GeV^2, $\Lambda = 0.46$ GeV, $\tilde{g} = 0.64$, $\mu' = 0.15$ GeV as a function of σ (in GeV-units) at $\pi_1 = 0$ – left picture, and of π_1 (in GeV) at $\sigma = 0$ – right picture.

condensation of quarks or pions takes place at low values of the effective coupling (35) of quarks $\tilde{g} < 1$. However, when the chromomagnetic background field is present, the pion (at $\mu' > 0$) or quark condesation (see the case $\mu' = 0$ of Section II) take place even for small values of $\tilde{g} < 1$ (recall that in our consideration the current quark mass m was taken to be zero). If $\tilde{g} > 1$, the role of the external chromomagnetic field H is to enhance the effect of pion condensation that appeares in the model at $\mu' > 0$ already at $H = 0$. For further investigations, it would be interesting to study the influence of an external chromomagnetic field (the gluon condensate) on the pion condensation phenomenon at nonzero current quark mass m as well as to consider the chromomagnetic catalysis effect with another combinations of a background field, different from (20).

V.   ACKNOWLEDGEMENTS

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