Pion condensation in the Gross-Neveu model with nonzero baryon and isospin chemical potentials

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The properties of the two-flavored Gross-Neveu model in the (1+1)-dimensional spacetime are investigated in the presence of the isospin μ_I as well as quark number μ chemical potentials both at zero and nonzero temperatures. The consideration is performed in the limit $N_c \to \infty$, i.e. in the case with an infinite number of colored quarks. In the plane of parameters μ_I, μ a rather rich phase structure is found, which contains phases with and without pion condensation. Moreover, we have found a great variety of one-quark excitations of these phases, including gapless and gapped quasiparticles.

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I. INTRODUCTION

During the last decade great attention was paid to the investigation of the QCD phase diagram in terms of baryonic as well as isotopic (isospin) chemical potentials. First of all, this interest is motivated by experiments on heavy-ion collisions, where we have to deal with dense baryonic matter which has an evident isospin asymmetry, i.e. different neutron and proton contents of initial ions. Moreover, the dense hadronic/quark matter inside compact stars is also isotopically asymmetric. Generally speaking, one of the important QCD applications is just to describe the dense and hot baryonic matter. However, in the above mentioned realistic situations the density is rather small, and weak coupling QCD analysis is not applicable. So, different nonperturbative methods or effective theories such as chiral effective Lagrangians and especially Nambu – Jona-Lasinio (NJL) type models [1] are usually employed for the consideration of the properties of dense and hot baryonic matter under heavy-ion experimental and/or compact star conditions, i.e. in the presence of such external conditions as temperature and chemical potentials, magnetic field, finite size effects etc (see, e.g., the papers [2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein). In particular, the color superconductivity [4, 5] as well as charged pion condensation [11, 12, 13, 14, 15, 16] phenomena of dense quark matter were investigated in the framework of these QCD-like effective models.

It is necessary to note that an effective description of QCD in terms of NJL models, i.e. through an employment of four-fermionic theories in (3+1)-dimensional spacetime, is usually valid only at *rather low* energies and densities. Besides, at present time there is the consensus that another class of theories, the set of (1+1)-dimensional Gross-Neveu (GN) type models [17, 18], can also be used for a good qualitative consideration of the QCD properties *without any restrictions* on the energy/density values, which is in an encouranging contrast with NJL models. Indeed, the GN type models are renormalizable, the asymptotic freedom and spontaneous chiral symmetry breaking are another properties inherent both for QCD and GN theories etc. In addition, the $\mu - T$ phase diagram is qualitatively the same in QCD and GN model [19, 20, 21, 22, 23] (here μ is the quark number chemical potential and T is the temperature). The GN type models are also very suitable for the description of physics in such quasi one-dimensional condensed matter systems as polyacetylene [24]. Due to the relative simplicity of GN models in the leading order of the large N_c-expansion (N_c is the number of colored quarks), their use is very convenient for the application of nonperturbative methods in quantum field theory [25]. Moreover, it is necessary to note that just in the leading order of the large N_c-expansion the well known no-go theorem by Mermin-Wagner-Coleman [26] apparently forbidding the spontaneous breaking of continuous symmetries in the considered (1+1)-dimensional models is not valid [21, 22, 23].

Thus, such phenomena of dense QCD as color superconductivity, where the color group is broken spontaneously, and charged pion condensation, where the spontaneous breaking of the continuous isospin symmetry takes place, might be simulated in terms of simpler (1+1)-dimensional GN type models (see, e.g., [22] and [27], correspondingly).

In our previous paper [27] the phase diagram of the (1+1)-dimensional GN model with two massless quark flavors was investigated under the constraint that quark matter occupies a finite space volume (see also the relevant papers [28]). In particular, there we have studied in the large N_c-limit the charged pion condensation phenomenon in cold quark matter with zero baryonic density, i.e. at $\mu = 0$, but nonzero isotopic density, i.e. with nonzero isospin chemical potential μ_I. In contrast, in the present paper in the leading order of the $1/N_c$-expansion we consider the phase portrait of the above mentioned GN model in a more general case, where both isospin- and quark number chemical potentials are nonzero, i.e. $\mu_I \neq 0$ and $\mu \neq 0$. Moreover, we deal with nonzero temperature and with spacetime of the usual topology, $R^1 \times R^1$. We hope that our investigations will shed some new light on the physics of dense and hot isotopically asymmetric quark matter which can be observed in heavy-ion collision experiments or might exist in compact stars, where baryon density is obviously nonzero (i.e. $\mu \neq 0$). Our consideration is based, for simplicity, on the approach with homogeneous condensates (an extension to inhomogeneous condensates was also considered in [23, 29]).

The paper is organized as follows. In Section II the thermodynamic potential of the two-flavored Gross-Neveu model is obtained in the presence of quark number as well as isotopic chemical potentials. In Section III and IV the phase structure of the model is investigated at zero and nonzero temperatures, correspondingly. It turns out that at zero temperature, $\mu_I \neq 0$ and rather small values of μ the gapped pion condensed phase occurs. However, at larger values of μ the dense quark matter phase with chirally broken phase and gapless quasiparticles is realized.

II. THE MODEL AND ITS THERMODYNAMIC POTENTIAL

We consider a (1+1)-dimensional model which describes dense quark matter with two massless quark flavors (u and d quarks). Its Lagrangian has the form

$$L = \bar{q}\Big[\gamma^\nu \mathrm{i}\partial_\nu + \mu\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0\Big]q + \frac{G}{N_c}\Big[(\bar{q}q)^2 + (\bar{q}\mathrm{i}\gamma^5\vec{\tau}q)^2\Big], \tag{1}$$

where the quark field $q(x) \equiv q_{i\alpha}(x)$ is a flavor doublet ($i = 1,2$ or $i = u,d$) and color N_c-plet ($\alpha = 1,...,N_c$) as well as a two-component Dirac spinor (the summation in (1) over flavor, color, and spinor indices is implied); τ_k ($k = 1,2,3$) are Pauli matrices; the quark number chemical potential μ in (1) is responsible for the nonzero baryonic density of quark matter, whereas the isospin chemical potential μ_I is taken into account in order to study properties of quark matter at nonzero isospin densities (in this case the densities of u and d quarks are different). Evidently, the model (1) is a generalization of the (1+1)-dimensional Gross-Neveu model [17] with a single massless quark color N_c-plet to the case of two quark flavors and additional chemical potentials. As a result, in the considered case we have a modified chiral symmetry group. Indeed, at $\mu_I = 0$ apart from the color SU(N_c) symmetry, the Lagrangian (1) is invariant under transformations from the chiral $SU_L(2) \times SU_R(2)$ group. However, at $\mu_I \neq 0$ this symmetry is reduced to $U_{I_3L}(1) \times U_{I_3R}(1)$, where $I_3 = \tau_3/2$ is the third component of the isospin operator (here and above the subscripts L, R mean that the corresponding group acts only on left, right handed spinors, respectively). Evidently, this symmetry can also be presented as $U_{I_3}(1) \times U_{AI_3}(1)$, where $U_{I_3}(1)$ is the isospin subgroup and $U_{AI_3}(1)$ is the axial isospin subgroup. Quarks are transformed under these subgroups as $q \to \exp(\mathrm{i}\alpha\tau_3)q$ and $q \to \exp(\mathrm{i}\alpha\gamma^5\tau_3)q$, respectively.

The linearized version of the Lagrangian (1), which contains composite bosonic fields $\sigma(x)$ and $\pi_a(x)$ ($a = 1,2,3$), has the following form:

$$\tilde{L} \;=\; \bar{q}\Big[\gamma^\nu \mathrm{i}\partial_\nu + \mu\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0 - \sigma - \mathrm{i}\gamma^5\pi_a\tau_a\Big]q - \frac{N_c}{4G}\Big[\sigma\sigma + \pi_a\pi_a\Big]. \tag{2}$$

From the Lagrangian (2) one gets the following constraint equations for the bosonic fields

$$\sigma(x) = -2\frac{G}{N_c}(\bar{q}q); \quad \pi_a(x) = -2\frac{G}{N_c}(\bar{q}\mathrm{i}\gamma^5\tau_a q). \tag{3}$$

Obviously, the Lagrangian (2) is equivalent to the Lagrangian (1) when using the constraint equations (3). Furthermore, it is clear from (3) that the bosonic fields transform under the isospin $U_{I_3}(1)$ and axial isospin $U_{AI_3}(1)$ subgroups in the following manner:

$$\begin{aligned} U_{I_3}(1):&\quad \sigma \to \sigma;\;\; \pi_3 \to \pi_3;\;\; \pi_1 \to \cos(2\alpha)\pi_1 + \sin(2\alpha)\pi_2;\;\; \pi_2 \to \cos(2\alpha)\pi_2 - \sin(2\alpha)\pi_1,\\ U_{AI_3}(1):&\quad \pi_1 \to \pi_1;\;\; \pi_2 \to \pi_2;\;\; \sigma \to \cos(2\alpha)\sigma + \sin(2\alpha)\pi_3;\;\; \pi_3 \to \cos(2\alpha)\pi_3 - \sin(2\alpha)\sigma. \end{aligned} \tag{4}$$

Due to the transformation rules (4) such quantity as the thermodynamic potential (TDP) depends effectively only on the two combinations $(\sigma^2 + \pi_3^2)$ and $(\pi_1^2 + \pi_2^2)$ of the bosonic fields, which are invariants with respect to the $U_{I_3}(1) \times U_{AI_3}(1)$ group. In this case, without loss of generality, one can put $\pi_2 = \pi_3 = 0$, and study the TDP as a function of only two variables, $M \equiv \sigma$ and $\Delta \equiv \pi_1$. Throughout the paper we suppose for simplicity that the condensates M and Δ are homogeneous quantities which do not depend on the space coordinate. In order to avoid the no-go theorem [26], which forbids the spontaneous breaking of continuous symmetries in the considered case of one space direction, all our considerations are performed in the leading order of the $1/N_c$-expansion, i.e. in the limit $N_c \to \infty$, where the TDP of the model looks like (see, e.g., [27])

$$\Omega_{\mu,\nu}(M,\Delta) = \frac{M^2 + \Delta^2}{4G} + \mathrm{i}\int \frac{d^2p}{(2\pi)^2} \ln\Big\{\Big[(p_0+\mu)^2 - (E_\Delta^+)^2\Big]\Big[(p_0+\mu)^2 - (E_\Delta^-)^2\Big]\Big\}, \tag{5}$$

where $E_\Delta^\pm = \sqrt{(E^\pm)^2 + \Delta^2}$, $E^\pm = E \pm \nu$, $\nu = \mu_I/2$ and $E = \sqrt{p_1^2 + M^2}$. It is clear that the TDP $\Omega_{\mu,\nu}(M,\Delta)$ is symmetric under the transformations $\mu \to -\mu$ and/or $\nu \to -\nu$. So it is sufficient to consider only the region $\mu \geq 0$,

$\nu \geq 0$. Taking into account this constraint as well as integrating in (5) over p_0, we obtain for the TDP of the system at zero temperature the following expression:

$$\Omega_{\mu,\nu}(M,\Delta)=\frac{M^2+\Delta^2}{4G}-\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\Big\{E_\Delta^+ + E_\Delta^- + (\mu-E_\Delta^+)\theta(\mu-E_\Delta^+)+(\mu-E_\Delta^-)\theta(\mu-E_\Delta^-)\Big\}, \tag{6}$$

where $\theta(x)$ is the Heaviside step function. Since we are going to study the phase diagram of the initial GN model, the system of gap equations is needed:

$$\begin{aligned}
0=\frac{\partial\Omega_{\mu,\nu}(M,\Delta)}{\partial M} &\equiv \frac{M}{2G}-M\int_{-\infty}^{\infty}\frac{dp_1}{2\pi E}\Big\{\frac{\theta(E_\Delta^+-\mu)E^+}{E_\Delta^+}+\frac{\theta(E_\Delta^--\mu)E^-}{E_\Delta^-}\Big\},\\
0=\frac{\partial\Omega_{\mu,\nu}(M,\Delta)}{\partial \Delta} &\equiv \frac{\Delta}{2G}-\Delta\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\Big\{\frac{\theta(E_\Delta^+-\mu)}{E_\Delta^+}+\frac{\theta(E_\Delta^--\mu)}{E_\Delta^-}\Big\}.
\end{aligned} \tag{7}$$

Evidently, the coordinates M and Δ of the global minimum point of the TDP (6) supply us with two order parameters (gaps), which are proportional to the ground state expectation values $\langle\bar{q}q\rangle$ and $\langle\bar{q}i\gamma^5\tau_1 q\rangle$, respectively. If only the gap M is nonzero, then in the ground state of the model the axial isospin symmetry $U_{AI_3}(1)$ (at $\mu_I\neq 0$) is spontaneously broken down. However, if only the gap $\Delta\neq 0$, then the ground state describes the phase with charged pion condensation, where the isospin $U_{I_3}(1)$ symmetry is spontaneously broken. Note that in this phase the space parity is also spontaneously broken.

III. PHASE PORTRAIT AT ZERO TEMPERATURE

First of all, let us consider the phase portrait of the initial GN model (1) using as a starting point the TDP (6). Since it is an ultraviolet divergent quantity, one should renormalize it, using a special dependence of the bare coupling constant $G\equiv G(\Lambda)$ on the cutoff parameter Λ (Λ restricts the integration region in the divergent integrals, $|p_1|<\Lambda$). In our previous paper [27] the following prescription for the bare coupling constant $G(\Lambda)$ was used,

$$\frac{1}{2G(\Lambda)}=\frac{1}{\pi}\int_{-\Lambda}^{\Lambda}dp_1\frac{1}{\sqrt{M_0^2+p_1^2}}=\frac{2}{\pi}\ln\left(\frac{\Lambda+\sqrt{M_0^2+\Lambda^2}}{M_0}\right), \tag{8}$$

where M_0 is the dynamically generated quark mass in the vacuum, i.e. at $\mu=0$ and $\mu_I=0$ (see below). Then, introducing the quantity $\Omega_{\mu,\nu}(M,\Delta;\Lambda)$,

$$\Omega_{\mu,\nu}(M,\Delta;\Lambda)=\frac{M^2+\Delta^2}{4G(\Lambda)}-\int_{-\Lambda}^{\Lambda}\frac{dp_1}{2\pi}\Big\{E_\Delta^+ + E_\Delta^- + (\mu-E_\Delta^+)\theta(\mu-E_\Delta^+)+(\mu-E_\Delta^-)\theta(\mu-E_\Delta^-)\Big\}+\frac{\Lambda^2}{\pi}, \tag{9}$$

it is possible to obtain the renormalized (finite) expression for the TDP:

$$\Omega_{\mu,\nu}(M,\Delta)=\lim_{\Lambda\to\infty}\Omega_{\mu,\nu}(M,\Delta;\Lambda). \tag{10}$$

(The renormalized expression for the gap equations is obtained in the limit $\Lambda\to\infty$, if the replacements $G\to G(\Lambda)$ and $|p_1|<\Lambda$ are done in (7), or by a direct differentiation of the expression (10).) In particular, at $\mu=0$ and $\mu_I=0$ we have from (10):

$$\Omega_{\mu,\nu}(M,\Delta)\Big|_{\mu=0,\nu=0}\equiv V_0(\sqrt{M^2+\Delta^2})=\frac{M^2+\Delta^2}{2\pi}\left[\ln\left(\frac{M^2+\Delta^2}{M_0^2}\right)-1\right]. \tag{11}$$

Since for a strongly interacting system the space parity is expected to be a conserved quantity in the vacuum, we put Δ equal to zero in (11). As a result, the global minimum of the TDP (11) (usually, this quantity is called effective potential in the vacuum) lies in the point $M=M_0$, which means that in the vacuum the dynamically generated quark mass is just the parameter M_0 introduced in (8). However, in the general case, i.e. at nonzero values of the chemical potentials, this quantity depends certainly on μ,μ_I and obeys the gap equations (7).

Numerical investigations show that local minima of the TDP (10) can occur only on the M- or Δ-axes (other solutions of the gap system (7) correspond to saddle points of the TDP (10)). So it is enough to restrict the consideration of the TDP (10) to the regions $M=0,\Delta\neq 0$ (Δ-axis) or $M\neq 0,\Delta=0$ (M-axis). Moreover, since the TDP is an even function with respect to the transformations $M\to -M$ or $\Delta\to-\Delta$, we will suppose further that

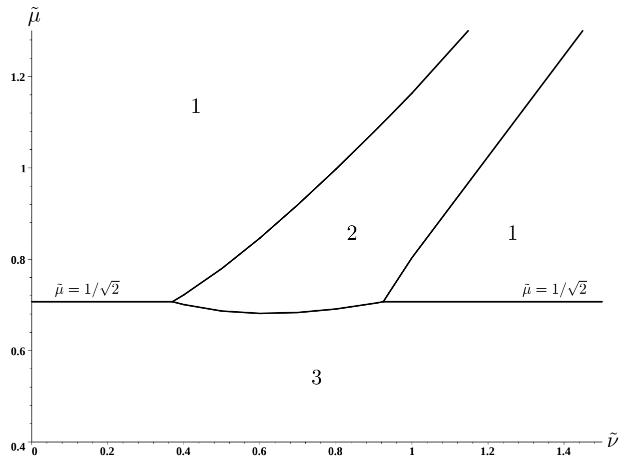

FIG. 1: The $(\tilde\mu, \tilde\nu)$ phase portrait of the model at $T = 0$ and $\tilde\nu > 0$. Here $\tilde\mu = \frac{\mu}{M_0}$, $\tilde\nu = \frac{\mu_I}{2M_0}$, and M_0 is the quark mass in the vacuum. Number 1 denotes the symmetric phase with massless quarks, number 2 – the normal quark matter phase with massive quarks, and 3 denotes the pion condensed phase.

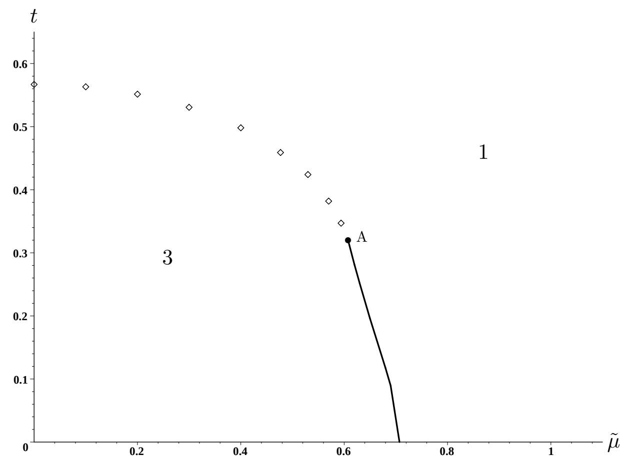

FIG. 2: The $(t, \tilde\mu)$ phase portrait of the model at $\mu_I = 0.2M_0$ ($t = \frac{T}{M_0}, \tilde\mu = \frac{\mu}{M_0}$). The solid and dotted lines are the critical curves of the 1st and 2nd order phase transitions, correspondingly. They are divided by the tricritical point A. Other notions are the same as in Fig. 1.

$M, \Delta \geq 0$. As a result we have

$$\Omega_{\mu,\nu}(M=0,\Delta) = V_0(\Delta) - \frac{\nu^2}{\pi} + \frac{\theta(\mu-\Delta)}{\pi}\left[\Delta^2 \ln\left(\frac{\mu+\sqrt{\mu^2-\Delta^2}}{\Delta}\right) - \mu\sqrt{\mu^2-\Delta^2}\right], \tag{12}$$

$$\begin{aligned}\Omega_{\mu,\nu}(M,\Delta=0) = V_0(M) &+ \frac{\theta(\mu+\nu-M)}{2\pi}\left[M^2 \ln\left(\frac{\mu+\nu+\sqrt{(\mu+\nu)^2-M^2}}{M}\right) - (\mu+\nu)\sqrt{(\mu+\nu)^2-M^2}\right] \\ &+ \frac{\theta(|\mu-\nu|-M)}{2\pi}\left[M^2 \ln\left(\frac{|\mu-\nu|+\sqrt{(\mu-\nu)^2-M^2}}{M}\right) \right. \\ &\left. - |\mu-\nu|\sqrt{(\mu-\nu)^2-M^2}\right]\Big(1-\theta(M-\nu)\theta(\nu-\mu)\Big),\end{aligned} \tag{13}$$

where the function $V_0(x)$ is presented in (11). Compairing the global minima of the TDPs (12)-(13), it is possible to obtain the global minimum point of the genuine TDP (10) of the model. Clearly, its form (and, as a result, the phase of the model) depends on the values of μ and ν. So, the total information about the behaviour of the global minimum point vs μ, ν can be presented by the phase portrait given in Fig. 1. (Note, it is valid only for $\nu > 0$ values. In the $\nu = 0$ case the phase structure is described below). There, in the corresponding $(\tilde\nu, \tilde\mu)$-plane (excluding the $\tilde\mu$-axis), where $\tilde\mu = \frac{\mu}{M_0}$, $\tilde\nu = \frac{\nu}{M_0} = \frac{\mu_I}{2M_0}$, you can see three respective phase regions denoted by the numbers 1, 2, 3. For the values of $(\tilde\nu, \tilde\mu)$ from the region 1, 2, and 3 the global minimum point of the TDP (10) has the form $(M = 0, \Delta = 0)$, $(M \neq 0, \Delta = 0)$, and $(M = 0, \Delta \neq 0)$, correspondingly. As a result, in the region 1 the chirally $U_{I_3}(1) \times U_{AI_3}(1)$-symmetric phase with massless quarks is arranged. In the region 2, where the order parameter M is nonzero, this symmetry is spontaneously broken down to the isospin $U_{I_3}(1)$ subgroup (in this region the order parameter M is a smooth function vs μ and ν). We call this phase the normal quark matter phase, since here quarks dynamically acquire a mass which is equal to the order parameter M, and space parity is not broken. Finally, the region 3 corresponds to the charged pion condensed phase, because of the nonzero order parameter Δ. Furthermore, for all points of this region $\Delta \equiv M_0$. Note that on the phase boundaries phase transitions of the first order occur.

It is worth to remark that in early papers (see e.g. [19, 20]), the phase structure of the GN model was studied in the exceptional case of $\nu = 0, \mu \neq 0$ with homogeneous condensates. As it was shown there, at $\mu > M_0/\sqrt{2}$ the symmetric phase 1 is observed. However, at $\mu < M_0/\sqrt{2}$ the normal quark matter phase is realized with a constant quark mass equal to M_0. [1] Let us denote this particular normal phase as phase 4, which is not depicted in Fig. 1. [2] There are

[1] If an inhomogeneous ansatz for condensates is taken into account, then in some GN phase diagrams an additional crystalline phase might appear at $\nu = 0$ and intermediate values of μ [23, 29]. The existence of the crystalline phase in the general case with $\nu \neq 0$ is not investigated up to now.

[2] We would like to stress once more that in this figure the phase 3 does not occupy the part of the $\tilde\mu$-axis ($\tilde\nu = 0$, $0 < \tilde\mu < 1/\sqrt{2}$), where actually the phase 4 is arranged.

two essential differences between phases 2 and 4. The first one manifests itself in their dynamical properties such as quasiparticle excitations. Recall that in the most general case the energy spectrum of the u-, d-, $\bar{u}$-, $\bar{d}$-quasiparticles (quarks) has the following form (see, e.g., [13]):

$$E_u(p_1)=E_\Delta^- -\mu,\quad E_d(p_1)=E_\Delta^+ -\mu,\quad E_{\bar{u}}(p_1)=E_\Delta^+ +\mu,\quad E_{\bar{d}}(p_1)=E_\Delta^- +\mu. \tag{14}$$

It is easily seen from (14) that in the phase 4, where $\nu=0$, $\Delta=0$, $M=M_0$ and $\mu<M_0/\sqrt{2}$, there is a gap in the quasiparticle energy spectrum, i.e. in the phase 4 all quasiparticles are gapped excitations. In contrast, in the phase 2 ($\nu\neq 0$) the situation is quite different and here in the energy spectrum of the u-quasiparticles the gap is absent. It means that for each point of the region 2 in Fig. 1 there exists a space momentum $p_1^\star$ such that $E_u(p_1^\star)=0$. (For example, the point ($\tilde{\nu}=0.7,\tilde{\mu}\approx 0.92$) is in the phase 2 (near its boundary), where $\Delta=0$ and $M\approx 0.31M_0$. In this case $p_1^\star\approx 1.59M_0$ etc.) So, in analogy with gapped and gapless color superconductivity [4], the phase 2 may be called gapless normal quark matter phase.

The second difference between phases 2 and 4 is a thermodynamic one and related with the density of quarks in both phases. It is clear that in the phase 4, i.e. at $\nu=0$, $M=M_0$ and $\mu<M_0/\sqrt{2}$, the TDP from (13) does not depend on the quark chemical potential μ. As a result, we see that in this phase the quark density $n_q\equiv-\partial\Omega/\partial\mu$ is equal to zero. However, in the phase 2 the density of quarks is not zero. So the consideration of dense quark matter in the framework of the GN model is more adequate in terms of both nonzero isotopic and quark number chemical potentials, $\nu\neq 0$ and $\mu\neq 0$, than in the simpler case with $\nu=0$, $\mu\neq 0$.

It is seen from Fig. 1 that the symmetric phase 1, where $M=0$ and $\Delta=0$, is a disconnected manifold consisting of two different parts. (Note that in the ground state of this phase the quark number density is not zero.) The first part is arranged above the phase 2 and corresponds to $\mu>\nu$, whereas in the second one, which is below the phase 2, we have $\mu<\nu$. It turns out that the dynamical properties of these two regions are different. Indeed, for the part of the phase 1 which is above the phase 2 both u- and d-quasiparticles are gapless. However, in the rest of the phase 1, which corresponds to $\mu<\nu$, only u-quasiparticles are gapless, as it is clear from the corresponding dispersion relations of these quasiparticles, $E_u(p_1)=|p_1-\nu|-\mu$ and $E_d(p_1)=p_1+\nu-\mu$, in the phase 1. Hence, such dynamical properties of dense quark matter as transport phenomena (e.g., conductivities etc) can occur in a qualitatively different way in these subphases of the symmetric phase 1.

Finally, note that all quasiparticle excitations of the pion condensed phase 3 are gapped ones. Indeed, it is clear from (14) that at $\Delta=M_0$, $M=0$ and $\mu\leq M_0/\sqrt{2}$ the quantities (14) do not vanish. So there is a gap in the energy spectrum of quasiparticles in the pion condensed phase 3.

IV. PHASE PORTRAIT AT NONZERO TEMPERATURES

Now let us stydy the influence of the temperature T on the phase structure of the considered GN model with two chemical potentials μ and $\nu\equiv\mu_I/2$. To get the corresponding thermodynamic potential $\Omega_{\mu,\nu,T}(M,\Delta)$, one can simply start from the expression for the TDP at zero temperature (5) and perform the following standard replacements:

$$\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}(\cdots)\to iT\sum_{n=-\infty}^{\infty}(\cdots),\qquad p_0\to p_{0n}\equiv i\omega_n\equiv i\pi T(2n+1),\qquad n=0,\pm1,\pm2,..., \tag{15}$$

i.e. the p_0-integration should be replaced in favour of the summation over an infinite set of Matsubara frequencies ω_n. Summing in the obtained expression over Matsubara frequencies (the corresponding technique is presented, e.g., in [30]), one can find for the TDP:

$$\begin{aligned}\Omega_{\mu,\mu_I,T}(M,\Delta) \;=\; &\frac{M^2+\Delta^2}{4G}-\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\Big\{E_\Delta^+ +E_\Delta^- +T\ln\left[1+e^{-\beta(E_\Delta^+-\mu)}\right]+T\ln\left[1+e^{-\beta(E_\Delta^++\mu)}\right]\\ &+\;T\ln\left[1+e^{-\beta(E_\Delta^--\mu)}\right]+T\ln\left[1+e^{-\beta(E_\Delta^-+\mu)}\right]\Big\},\end{aligned} \tag{16}$$

where $\beta=1/T$. As in the case with zero temperature, the renormalized expression for $\Omega_{\mu,\nu,T}(M,\Delta)$ can be obtained by the replacement $G\to G(\Lambda)$ (see formula (8)) which, along with the cutting of the integration region in (16), $|p_1|<\Lambda$, leads in the limit $\Lambda\to\infty$ to the finite expression for the TDP. Numerical investigations show that all possible local minima of the obtained TDP lie on the M- and Δ-axis. So it is sufficient to deal with corresponding restrictions of the TDP on these axes, i.e. with the following functions,

$$\Omega_{\mu,\mu_I,T}(M=0,\Delta) \;=\; V_0(\Delta)-\frac{\nu^2}{\pi}-\frac{2T}{\pi}\int_0^\infty dp_1\ln\Big\{\left[1+e^{-\beta(\mathcal{E}-\mu)}\right]\left[1+e^{-\beta(\mathcal{E}+\mu)}\right]\Big\}, \tag{17}$$

$$\begin{aligned}\Omega_{\mu,\mu_I,T}(M,\Delta=0) \;=\; &V_0(M)-\frac{T}{\pi}\int_0^\infty dp_1\ln\Big\{\left[1+e^{-\beta(E+\nu-\mu)}\right]\left[1+e^{-\beta(E+\nu+\mu)}\right]\Big\}\\ &-\frac{T}{\pi}\int_0^\infty dp_1\ln\Big\{\left[1+e^{-\beta(E-\nu-\mu)}\right]\left[1+e^{-\beta(E-\nu+\mu)}\right]\Big\},\end{aligned} \tag{18}$$

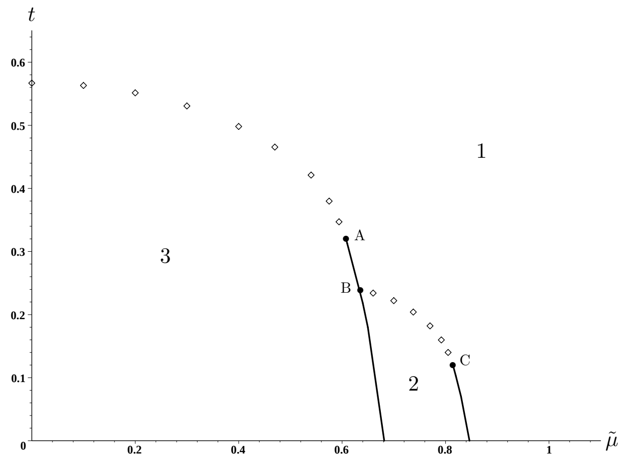

FIG. 3: The $(t, \tilde{\mu})$ phase portrait of the model at $\mu_I = 1.2M_0$ ($t = \frac{T}{M_0}$, $\tilde{\mu} = \frac{\mu}{M_0}$). The points A, B, C are tricritical points. Other notions are the same as in Figs. 1, 2.

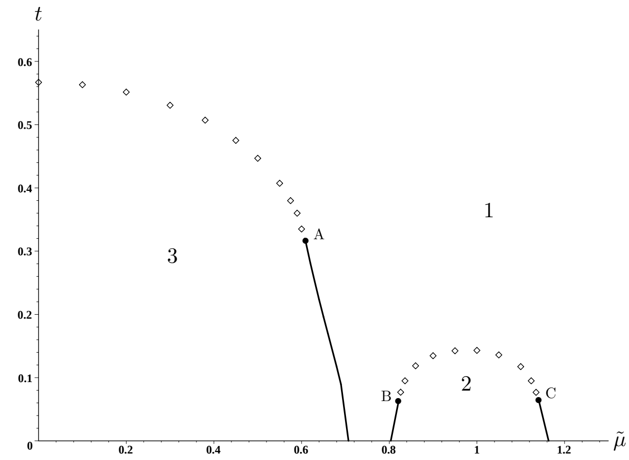

FIG. 4: The $(t, \tilde{\mu})$ phase portrait of the model at $\mu_I = 2M_0$ ($t = \frac{T}{M_0}$, $\tilde{\mu} = \frac{\mu}{M_0}$). Other notions are the same as in the previous figures.

where the effective potential $V_0(x)$ is given in (11), $E = \sqrt{p_1^2 + M^2}$, and $\mathcal{E} = \sqrt{p_1^2 + \Delta^2}$. Comparing the global minima of the functions (17) and (18), it is possible to establish the global minimum point of the renormalized TDP (16). Then, the dependence of the global minimum point vs T, μ, ν defines the phase structure of the model.

Using this prescription in our numerical investigations of the TDPs (17)-(18), we have found the three (μ, T)-phase portraits of the initial GN model, depicted in Figs. 2, 3, 4, for three qualitatively different values of the isospin chemical potentials, $\mu_I = 0.2M_0$, $\mu_I = 1.2M_0$, and $\mu_I = 2M_0$, respectively. There, the solid and dotted lines correspond to curves of first- and second order phase transitions. Moreover, there are several tricritical points, A, B, C, in these phase diagrams. (A point of the phase diagram is called a tricritical one, if an arbitrary small vicinity of it contains both first- and second order phase transition points.)

V. SUMMARY AND CONCLUSIONS

Recent investigations of the phase diagram of isotopically asymmetric dense quark matter in terms of NJL models show that their pion condensation content is not yet fully understood. Indeed, the number of the charged pion condensation phases of the phase diagram depends strictly on the parametrization set of the NJL model. It means that for different values of the coupling constant, cutoff parameter etc just the same NJL model predicts different numbers of pion condensation phases of quark matter both with or without electric neutrality constraint (see, e.g., [13, 15]). So to obtain a more objective information about the pion condensation phenomenon of dense quark matter it is important to invoke alternative approaches. One of them, which qualitatively quite successfully imitates the QCD properties (see also the Introduction), is based on the consideration of this phenomenon in terms of the leading order of the large N_c-technique in the framework of (1+1)-dimensional GN models.

In the present paper we have studied the phase structure of the GN model (1) in terms of temperature, quark number (μ)- as well as isospin (μ_I) chemical potentials in the limit $N_c \to \infty$. Firstly, we have found that at $T = 0$ the charged pion condensation phase of the GN model is realized inside the chemical potential region $\mu_I > 0$, and μ is not greater than $M_0/\sqrt{2}$ (see Fig. 1). The corresponding quasiparticle excitations have a gap in the energy spectrum. In contrast, in the NJL phase diagram the pion condensation phases occupy a compact region and for some parameterization schemes the gapless pion condensation might occur [12, 13, 14, 15].

Secondly, at $\mu_I > 0$ and rather large values of the quark number chemical potential μ we have found the normal quark matter phase 2 (see Fig. 1) in which chiral symmetry is broken spontaneously. Moreover, some of its one-quark excitations are gapless quasiparticles, and the quark number density in the ground state is not zero. (In comparision, in the early investigations of the GN model at $\mu_I = 0$ and $\mu \neq 0$ the chirally broken phase was also found, but with zero quark number density and gapped quasiparticles [19].) The region corresponding to this phase in the (μ_I, μ)-plane is not restricted. In contrast, in the chirally symmetric NJL model, i.e. with zero current quark mass, the (μ_I, μ)-phase diagram does not contain the normal quark matter phase with such properties (i.e. with nonzero quark number density, gapless quasiparticles, and broken chiral invariance), at least at $\mu_I > 0$ [13].

Thirdly, we have found a rather rich quasiparticle excitation spectrum for the chirally symmetric phase 1. In this phase, if $\mu > \nu$, all quasiparticles are gapless. However, if $\mu < \nu$, then the gap is absent in the spectrum of only u-quasiparticles.

We hope that our investigation of the GN phase diagram will shed some new light on the phase structure of QCD at nonzero baryonic and isotopic densities. Thus, even in the most simple approach to the GN phase diagram we have found a variety of phases with rather rich dynamical contents. Obviously, a more realistic imitation of the QCD phase diagram requires to include also a nonzero bare quark mass, i.e. to study massive GN models, as well as to take into account the possibility of inhomogeneous condensates.

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