Does there arise a significant enhancement of the dynamical quark mass in external magnetic field?

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Recently, it was found in QED that the generation of a dynamical electron mass in a strong magnetic field is significantly enhanced by the perturbative electron mass. In the present paper, the related question of a possible enhancement of the dynamical quark mass in an external magnetic field and with a bare mass term is investigated in the Nambu–Jona-Lasinio model.

In the recent paper [1] some aspects of the well-known magnetic catalysis effect were studied in quantum electrodynamics (QED). In particular, it was shown that an enormously high external magnetic field B_{m_e} of the order of 10^{82} G would be needed to create dynamically the common experimental value $m_e \approx 0.5$ MeV of the electron mass m_e in the massless QED. At the same time, if the bare mass of the theory is nonvanishing and corresponds to the experimental value m_e, then in the presence of an external magnetic field of the same value B_{m_e} the dynamical mass of an electron is enhanced to the value almost ten times larger than m_e. (The behaviour of the dynamical electron mass in a strong magnetic field in massive QED was also considered previously [2].) As it was claimed in [1], such a significant enhancement of the dynamical electron mass in QED is a new effect that can find applications in astrophysics and cosmology, and it deserves to be investigated in more detail, and especially also in QCD. In particular, as it was predicted in [1], in the case with much smaller and realistic magnetic field values around 10^{15} G, the typical magnetic fields of compact stars, a few percent measurable increase in the dynamical electron mass still exists.

It has been known that the magnetic catalysis effect, i.e. the spontaneous breaking of the chiral symmetry induced by an external magnetic field B, is an universal phenomenon, which takes place in different physical models (see, e.g., the reviews [3] as well as the original papers [4, 5, 6, 7, 8, 9, 10] and references therein). Thus, the following natural question arises: **Does it mean that the enhancement effect has an universal character as well?** In this paper, we study this problem in the framework of the Nambu – Jona-Lasinio (NJL) model with two quark flavors.

In four-dimensional spacetime and at $B = 0$ the system is described by the following Lagrangian:

$$\mathcal{L} = \bar{q}[i\gamma^\nu \partial_\nu - m_0]q + G[(\bar{q}q)^2 + (\bar{q}i\gamma^5\vec{\tau}q)^2], \tag{1}$$

where the quark field $q \equiv q_{i\alpha}$ is a flavor doublet ($i = 1, 2$ or $i = u, d$) and a color triplet ($\alpha = 1, 2, 3$ or $\alpha = r, g, b$) as well as a four-component Dirac spinor; τ_a stands for the Pauli matrices. It is supposed here that up and down quarks have an equal current (bare) mass m_0. Clearly, at $m_0 = 0$ this Lagrangian is invariant under the continuous chiral $SU(2)_L \times SU(2)_R$ group as well as under the discrete chiral transformation, $q \to i\gamma^5 q$. At the tree level, the Lagrangian (1) contains two free model parameters, the coupling constant G and the bare quark mass m_0. However, when including quantum effects (quark loops), one should regularize the corresponding loop integrals, for example, by cutting off the three-dimensional momentum space, i.e. supposing that $|\vec{p}| \le \Lambda$. Thus an additional free parameter, the cutoff Λ, appears in the model. In the mean field approximation the effective potential of the model (1) looks like (see, e.g., [11, 12])

$$V(m) = \frac{(m - m_0)^2}{4G} - \frac{3}{4\pi^2}\left[\Lambda(2\Lambda^2 + m^2)\sqrt{m^2 + \Lambda^2} - m^4 \ln\left(\frac{\Lambda + \sqrt{m^2 + \Lambda^2}}{m}\right)\right], \tag{2}$$

where m is the dynamical quark mass, which is connected with the bare mass m_0 and the vacuum expectation value of quark fields $\langle\bar{q}q\rangle$ through the relation

$$m = m_0 - 2G\langle\bar{q}q\rangle.$$

Note that it depends on the model parameters G, m_0, Λ and is determined by the gap equation

$$\frac{\partial}{\partial m}V(m) \equiv \frac{m - m_0}{2G} - \frac{3m}{\pi^2}\left[\Lambda\sqrt{m^2 + \Lambda^2} - m^2 \ln\left(\frac{\Lambda + \sqrt{m^2 + \Lambda^2}}{m}\right)\right] = 0. \tag{3}$$

Evidently [11, 12], at $m_0 = 0$ the dynamical quark mass m is a nonzero quantity only at $G > G_{crit} = \pi^2/(6\Lambda^2)$ (in this case the chiral symmetry of the model is spontaneously broken down). However, it follows from (3) that at $m_0 = 0$ and $G < G_{crit}$ we have $m \equiv 0$, and the chiral symmetry remains intact in this case. If $m_0 \ne 0$, then $m \ne 0$ for arbitrary values of G. Below, one can find some values of m vs m_0 in the second line of Tables I, II for $G < G_{crit}$.

The influence of an external constant and homogeneous magnetic field B on the properties of the NJL-type models was already considered in refs. [12, 13, 14, 15]. To obtain the corresponding Lagrangian, it is necessary to perform

in (1) the following replacement: $\partial_\nu \to \partial_\nu + iQA_\nu$, where A_ν is a vector-potential of an external magnetic field B, and $Q = \mathrm{diag}(e_1, e_2)$ is the electric charge matrix of quarks. Here $e_1 = 2|e|/3$ and $e_2 = -|e|/3$ (e is the electric charge of electrons) are the electric charges of u- and d- quarks, respectively. At $m_0 = 0$, the resulting Lagrangian is still symmetric with respect to the discrete symmetry $q \to i\gamma^5 q$, but it is no more invariant under the continuous chiral symmetry $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ because of the difference in quark electric charges. Clearly, at $B \neq 0$ the effective potential of the model is also modified (see, e.g., [12]) and looks like

$$V_{\mathrm{eff}}(M; B) = V(M) - \sum_{i=1}^{2} \frac{3(e_i H)^2}{2\pi^2} \left\{ \zeta'(-1, x_i) - \frac{1}{2}[x_i^2 - x_i]\ln x_i + \frac{x_i^2}{4} \right\}, \tag{4}$$

where $x_i = M^2/(2|e_i B|)$ for each $i = 1, 2$, $\zeta'(-1, x) = d\zeta(\nu, x)/d\nu|_{\nu=-1}$ ($\zeta(\nu, x)$ is the generalized Riemann zeta-function [16]), and $V(M)$ is the effective potential (2) with m replaced by M. The quantity $M = M(m_0, B)$ in (4) is the dynamical quark mass which is the solution of the gap equation

$$\frac{\partial}{\partial M} V_{\mathrm{eff}}(M; B) \equiv \frac{\partial}{\partial M} V(M) - I_1(M) - I_2(M) = 0, \tag{5}$$

where

$$I_i(M) = \frac{3M|e_i B|}{2\pi^2} \left\{ \ln \Gamma(x_i) - \frac{1}{2}\ln(2\pi) + x_i - \frac{1}{2}(2x_i - 1)\ln x_i \right\} \quad (i = 1, 2) \tag{6}$$

and $\Gamma(x)$ is the Euler gamma-function [16]. In what follows, we suppose that B is a nonnegative quantity, and take into consideration that $M(m_0, 0) = m$.

At zero bare mass $m_0 = 0$ the influence of an external magnetic field on the phase structure of the model (1) was already investigated, e.g., in [12, 13, 14, 15]. In particular, it was shown there that at $G < G_{crit}$ and $B = 0$ the global minimum of the effective potential (2) lies at the point $m = 0$, so that the chiral symmetries, both continuous and discrete, are not broken down. However, at arbitrary small values of B and $G < G_{crit}$ the global minimum of the effective potential (4) of the system is shifted to a nontrivial point. As a result, in this case the spontaneous breaking of the discrete chiral symmetry [1] is induced by an external magnetic field $B \neq 0$ (magnetic catalysis effect). Moreover, a dynamical quark mass $M \equiv M(m_0 = 0, B)$, which is the solution of the equation (5) at $m_0 = 0$, is also generated. Note, at $G > G_{crit}$ and $m_0 = 0$ the dynamical chiral symmetry breaking in the NJL model takes place even at $B = 0$ due to a rather strong interaction in the quark-antiquark channel.

Now, we have at our disposal all necessary formulas in order to solve the question raised at the beginning of the paper. Clearly, the two different possibilities should be studied, $G < G_{crit}$ and $G > G_{crit}$.

The weak coupling regime ($G < G_{crit}$). Since in this case, as in QED, the magnetic catalysis effect takes place in the NJL model, i.e. at $m_0 = 0$ a nonzero dynamical quark mass is induced by an external magnetic field, we are going to proceed in the spirit of the paper [1]. For simplicity, let us put $\Lambda = 1$ GeV, i.e. $G_{crit} \approx 1.65$ GeV^{-2}, and consider, for illustrations, two values of the coupling constant, $G = 0.5$ GeV^{-2} and $G = 1$ GeV^{-2}, which are smaller, than G_{crit}. Now, in order to compare our results with those of Wang, it is convenient to divide, as in [1], the numerical calculations into several stages.

Table I: The case $G = 0.5$ GeV^{-2}.

m_0 [GeV]	0	0.0003	0.003	0.03	0.3
m [GeV]	0	0.00043	0.0043	0.043	0.4
$2|e|B_m$ [GeV^2]	0	1.38	2.01	3.59	10.24
$M(m_0, B_m)$ [GeV]	0	0.00194	0.0154	0.113	0.74
$R = M(m_0, B_m)/m$		4.51	3.58	2.64	1.85

Table II: The case $G = 1$ GeV^{-2}.

m_0 [GeV]	0	0.0002	0.002	0.02	0.2
m [GeV]	0	0.0005	0.0051	0.051	0.4
$2|e|B_m$ [GeV^2]	0	0.435	0.67	1.36	4.86
$M(m_0, B_m)$ [GeV]	0	0.00212	0.0165	0.116	0.65
$R = M(m_0, B_m)/m$		4.16	3.24	2.30	1.64

(i) First, we put $B = 0$ and find the dynamical quark mass m, by solving equation (3) for different values of the bare mass m_0. For some representative values of m_0 (see the first line in Tables I, II) the corresponding values of m are presented in Tables I, II (see the second line there).

(ii) Next, one should find such a value of the magnetic field B_m, for which the solution of equation (5) at $m_0 = 0$ coincides with m, i.e. $M(m_0 = 0, B_m) = m$. For the values of m from Tables I, II the corresponding values of B_m are presented in the third line of Tables I, II. Since $\Lambda^2/|e| \approx 3$ GeV$^2 \approx 1.6 \cdot 10^{20}$ G, we see that in the NJL model the values

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[1] The continuous chiral symmetry remains to be broken due to the presence of the isospin-violating electric charge matrix Q in the covariant derivative of the modified Lagrangian (see the remarks above (4)).

of the quantity B_m have the order of magnitude of the NJL characteristic magnetic scale B_c, $B_c \equiv \Lambda^2/|e|$. In contrast, in QED (see [1]) the values of the corresponding quantity B_{m_e} are unrealistically higher than the characteristic QED magnetic scale, the Schwinger magnetic field $m_e^2/|e| \approx 4.4 \cdot 10^{13}$ G.

(iii) Finally, for each fixed value of m_0 and B_m we have solved the gap equation (5) and found the corresponding dynamical quark mass $M(m_0, B_m)$ as well as the ratio $R \equiv M(m_0, B_m)/m$ (these quantities are given in the fourth and fifth lines of Tables I, II respectively) which in some sense might serve as a measure of the dynamical quark mass enhancement effect [1]. It is seen from these tables that for the considered values of m_0 we have obtained $R < 5$ in the framework of the NJL model (1), and even $R < 2$ for the physically interesting case of a dynamical quark mass $m = 0.4$ GeV (with bare quark mass $m_0 = 0.3$ GeV for $G = 0.5$ GeV^{-2} or $m_0 = 0.2$ GeV for $G = 1$ GeV^{-2}). For comparision, let us quote the value $R \approx 10$, which was obtained in the same manner in QED [1].

Judging about the possibility of the enhancement effect in terms of the quantity R only, one might conclude that in the framework of the NJL model with a rather weak interaction, $G < G_{crit}$, the generation of a dynamical quark mass in a strong magnetic field is still enhanced at nonzero bare quark mass (since $R \approx 2$ in physically reliable cases with $m \approx 0.4$ GeV). But here this effect is not so pronounced as in QED, where $R \approx 10$. On the other hand, one should keep in mind that in the NJL model the enhancement of the dynamical quark mass takes place only at sufficiently high magnetic fields $B \gtrsim B_m \sim 10^{20}$ G. Indeed, our calculations show that in a more interesting case with realistic values of an external magnetic field $B \lesssim B_{phys} \equiv 10^{15}$ G, which are typical values of magnetic fields on the surface of young neutron stars, the dynamical quark mass $M(m_0 \neq 0, B)$ is with great accuracy equal to the dynamical (constituent) quark mass $m = M(m_0 \neq 0, 0)$ at $B = 0$. As a result, we see that for realistic values of B the enhancement effect is absent. In contrast, in QED such magnetic fields provide a few percent increase of the dynamical electron mass in comparision with m_e at $B = 0$, which is sufficient for observation of this effect in experiments [1]. To better understand the absence of the enhancement effect in the NJL model at $B \lesssim B_{phys}$, one should take into account that in the framework of the NJL model the field B_{phys} is comparatively weak, since $B_{phys} \ll B_c$, while in QED the field B_{phys} is comparatively strong, since $B_{phys} \gg m_e^2/|e|$, i.e. it is much greater than the Schwinger field.

The strong coupling regime ($G > G_{crit}$). Since NJL models are considered to be effective theories for low energy QCD only at $G > G_{crit}$, we have studied the influence of an external magnetic field B on the dynamical quark mass also in this case. At $G > G_{crit}$ and $B = 0$, the values of the NJL model parameters can be fixed through fitting of experimental data, and the typical set of Λ, m_0, G looks like [11]: $\Lambda = 0.6$ GeV, $m_0 = 0.005$ GeV, $G = 6.73$ GeV^{-2}, which corresponds to $G_{crit} \approx 4.57$ GeV^{-2} and the dynamical quark mass $M(m_0, B = 0) \approx 0.4$ GeV.

Now, using the gap equation (5), it is possible to conclude that at the value of the external magnetic field $B = B_c \equiv \Lambda^2/|e| \approx 6.4 \cdot 10^{19}$ G, which is the characteristic magnetic scale of the model for the above chosen value of $\Lambda = 0.6$ GeV, the dynamical quark mass $M(m_0, B_c)$ exceeds the dynamical quark mass $M(m_0, B = 0)$ no more than by 20%. At $B \approx 25B_c$, the corresponding dynamical quark mass is ten times larger than $M(m_0, B = 0)$, etc. Hence, at $G > G_{crit}$, and in a rather strong external magnetic field $B \gtrsim B_c$ the enhancement of the dynamical quark mass also takes place.

However, for values of a magnetic field B smaller than B_c, the excess of $M(m_0, B)$ over $M(m_0, B = 0)$ sharply decreases. (Note, that it is just in the region $B \lesssim B_c$ that the dynamics of QCD is qualitatively similar to that in the NJL model [17].) Indeed, at $B = 0.1B_c$ it is equal to 0.3% etc., and for the value $B = B_{phys} \equiv 10^{15}$ G (the field on the surface of young neutron stars) the difference between $M(m_0, B_{phys})$ and $M(m_0, B = 0)$ starts from the 10-th significant digit, i.e. it is vanishingly small. Therefore, for sufficiently small $B \lesssim B_{phys} \ll B_c$ the enhancement effect is absent both in the NJL model and QCD [2], and hence, in physical applications one might ignore the dependence of the dynamical quark mass on an external magnetic field B in this range. In spite of this fact, there are other phenomena, which can be observed at $B \lesssim B_{phys}$ in dense quark matter. Among them are the magnetic oscillation effect and other effects [12, 14] that are not directly related to the behaviour of the dynamical quark mass vs B. They are connected mostly with the thermodynamical properties of the system.

We remark in conclusion, that in order to answer the question raised at the beginning of the paper, one should first establish the ranges for the external magnetic field B under consideration. Then, if B varies in a certain vicinity of $B_{phys} \equiv 10^{15}$ G, the considered enhancement effect is intrinsic to QED (since here the magnetic field B_{phys} can be considered to be strong enough). At the same time, in QCD or the NJL model, the dynamical quark mass is not influenced by these realistic values of an external magnetic field. However, if B is rather strong, i.e., $B \gtrsim B_c$, the enhancement of a dynamical fermion mass does occur in QED and the NJL-type models, etc.

Note that in QCD in a strong magnetic field the situation with the enhancement effect might be very involved. Indeed, as it was shown in [17] in the chiral limit of QCD, the dynamical quark mass at $B \gtrsim B_c$ behaves quite unexpectedly since in a wide range of strong magnetic fields it is suppressed in comparision with the dynamical quark mass at $B = 0$ (one of the reasons is that QCD is an asymptotically free model). The same behaviour of the dynamical quark mass might be inherent to QCD with nonzero bare quark mass.

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[2] The behaviour of the quark condensate $\Sigma(B)$ at small values of magnetic fields $B \ll B_c$ was also considered in the framework of the chiral effective theory [18]. It is easily seen that in this case at $B \lesssim B_{phys}$ the quark condensate $\Sigma(B)$ exceeds $\Sigma(0)$ also in sufficiently small fractions of a percent and, hence, the dependence of chiral condensate on B might not be allowed for.

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