Chiral density waves in the NJL_2 model with quark number and isospin chemical potentials

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We investigate the phase portrait of the (1 + 1)-dimensional massless two-flavored NJL_2 model containing a quark number chemical potential μ and an isospin chemical potential μ_I in the limit of a large number of colors $N_c \to \infty$. Particular attention is paid to the question of to what extent the inclusion of an isospin asymmetry affects chiral condensates to have a spatial inhomogeneity in the form of the so-called chiral density waves (CDW) (chiral spirals). It is shown that, at zero temperature and comparatively small values of μ, i.e. at $\mu < \mu_c \approx 0.68M_0$ (M_0 is the dynamical quark mass in the vacuum), only the homogeneous charged pion condensation phase is realized for arbitrary nonzero values of μ_I. Contrary to this, for large values of $\mu > \mu_c$, two CDW phases appear in the (μ_I, μ)-phase diagram of the model. In the first phase, CDWs are clockwise twisted chiral spirals, and in the second phase they are counter-clockwise. The influence of nonzero temperature on the formation of the CDW phases is also investigated.

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I. INTRODUCTION

During the last decade, much attention has been attracted to the investigation of the QCD phase diagram in terms of quark number as well as isospin chemical potentials. First of all, this is motivated by heavy-ion collision experiments where dense baryonic matter has an evident isospin asymmetry, i.e. different neutron and proton contents of initial ions. Moreover, the dense hadronic/quark matter inside compact stars is also expected to be isotopically asymmetric. Generally speaking, it is understood that one of the important challenges for QCD is to describe the dense and hot baryonic matter in different physical situations. However, in the above-mentioned realistic situations, the quark density is rather small, and weak coupling QCD analysis is not applicable. So, different nonperturbative methods or effective theories such as chiral effective Lagrangians and especially Nambu–Jona-Lasinio (NJL) type models [1] are usually employed for the consideration of the properties of dense and hot baryonic matter under the conditions of heavy-ion experiments or in the compact star's interior, i.e. in the presence of external factors such as temperature, chemical potentials, magnetic field, finite size effects, etc. (see, e.g., [2–8] and references therein). In particular, phenomena of dense quark matter like color superconductivity [2,4–6] as well as charged pion condensation [9–11] were investigated in the framework of these QCD-like effective models.

It should be noted that an effective description of QCD in terms of NJL models, i.e. through an employment of four-fermionic theories in (3 + 1)-dimensional spacetime, is usually valid only at *comparatively low* energies and densities. At the same time, (1 + 1)-dimensional Gross-Neveu (GN) type models [12,13] are valid also at high energy and density, and due to their properties of renormalizability, asymptotic freedom and spontaneous chiral symmetry breaking, can also be used for a reasonable qualitative modeling of QCD even at finite temperature and hadron density [14–18]. Because of the relative simplicity of GN models in the leading order of the large N_c-expansion (N_c is the number of colored quarks), their use is convenient for the application of nonperturbative methods in quantum field theory [19]. Moreover, it is worth noting that it is in the leading order of the large N_c-expansion that the well-known no-go theorem of Mermin-Wagner-Coleman [20], apparently forbidding the spontaneous breaking of continuous symmetries in the (1 + 1)-dimensional models, becomes invalid [15–18]. (It means that, in the large N_c limit quantum fluctuations, which would otherwise destroy a long-range order corresponding to a spontaneous symmetry breaking, are suppressed by $1/N_c$ factors.) Note also that GN type models are quite suitable for the description of physics in quasi–one-dimensional condensed matter systems such as polyacetylene [21].

Thus, such phenomena of dense QCD as color superconductivity, where the color group is broken spontaneously, and charged pion condensation, where spontaneous breaking of the continuous isospin symmetry takes place, might be modeled in terms of renormalizable (1 + 1)-dimensional GN type models (see, e.g., [16,17,22–24], respectively).

In our previous papers [22–24], the phase diagram of a (1 + 1)-dimensional $SU_L(2) \times SU_R(2)$ symmetric NJL model[1] with two massless or massive quark flavors was

[1]In this paper we shall use the notation NJL model for theories with four-fermionic interactions also for (1 + 1)-dimensional models with *a continuous chiral symmetry group* instead of "chiral 2D GN model" due to the fact that the chiral structure of the Lagrangian is indeed closely related to the (3 + 1)-dimensional NJL model.

investigated in the leading order of the $1/N_c$-expansion and in the presence of the quark number as well as isospin chemical potentials. There we considered the case of order parameters (condensates) that are homogeneous, i.e. independent of the space coordinate. The situation corresponds to the conserved Lorentz and spatial translational invariance and is adequate to physical systems in vacuum, i.e. at zero chemical potentials. In dense baryonic matter, i.e. at nonzero quark number chemical potential, there might appear new phases with a spatially inhomogeneous chiral condensate which destroys both chiral and spatial translational invariance of the system. (See the relevant discussions made in the framework of both (1 + 1)-dimensional [18,25,26] and (3 + 1)-dimensional [27–31] models.) Thus, in this paper and in contrast to [22–24], we consider the phase portrait of the above-mentioned massless $SU_L(2)\times SU_R(2)$ symmetric NJL model with two chemical potentials in the leading order of the $1/N_c$-expansion, taking into account the possibility that the chiral condensate might become inhomogeneous and take the form of a (dual) chiral density wave (CDW). In this case the scalar quark-antiquark condensate, $\langle\bar{q}q\rangle$, and the pseudoscalar condensate of neutral π^0 mesons, $\langle\bar{q}\gamma^5\tau_3 q\rangle$, form a chiral spiral, i.e.

$$\langle\bar{q}q\rangle\sim\cos 2bx,\qquad\langle\bar{q}\gamma^5\tau_3 q\rangle\sim\sin 2bx,$$

where τ_3 is the isospin Pauli matrix, x is the space coordinate, and b is a wave vector which has to be determined dynamically through the thermodynamic potential. It is necessary to point out that the inhomogeneous CDW condensate is relevant to dense quark matter [32] and the chiral magnetic effect [33]. Both phenomena probably might be observed in heavy-ion collision experiments, where isotopic asymmetry is an inevitable property. So, we believe that the investigation of CDW condensates in the framework of the two-dimensional NJL model with isospin chemical potential could shed some light on the physics of heavy-ion collisions. It should be noted, however, that our investigation of a condensate inhomogeneity in the form of the chiral density wave is only a first step. There may exist, at least at zero isospin chemical potential, other more preferable spatially nonuniform ground state configurations of the chiral condensate like, for instance, chiral crystals [25,26,29] (in the last case, only the scalar $\langle\bar{q}q\rangle$ condensate is an inhomogeneous quantity, and other condensates are homogeneous ones), but, in general, they are much harder to deal with. For technical reasons, in studying CDW configurations we do not take into account a nonzero bare (current) quark mass, although recently some efforts to get rid of this assumption have been made [30].

The paper is organized as follows. In Sec. II, we derive in the leading order of the large N_c-expansion the expression for the thermodynamic potential of the $SU_L(2)\times SU_R(2)$ symmetric massless NJL_2 model with quark number chemical potential μ and isospin chemical potential μ_I at zero temperature. Here we also consider the possibility of a spatial inhomogeneity for the chiral condensates in the form of the so-called chiral density waves. First, the phase portrait of the model is discussed in the simple case of spatially homogeneous condensates in Sec. III, and then, in Sec. IV, the phase structure of the model in terms of μ and μ_I and at zero temperature is investigated for CDW inhomogeneous phases. The influence of nonzero temperature on the formation of CDW phases is considered in Sec. V. Finally, Sec. VI presents some concluding remarks.

II. THE MODEL AND ITS EFFECTIVE ACTION

We consider a (1 + 1)-dimensional NJL-type model with two massless quark flavors (u and d quarks) to mimic properties of real dense quark matter. Its Lagrangian has the form

$$\mathcal{L}=\bar{q}\Big[\gamma^\rho i\partial_\rho+\mu\gamma^0+\frac{\mu_I}{2}\tau_3\gamma^0\Big]q+\frac{G}{N_c}[(\bar{q}q)^2+(\bar{q}i\gamma^5\vec{\tau}q)^2],\qquad(1)$$

where our choice for the gamma matrices in (1 + 1)-dimensions is as follows: $\gamma^0=\sigma_2$, $\gamma^1=i\sigma_1$, $\gamma^5=\gamma^0\gamma^1=\sigma_3$, and the quark field $q(x)\equiv q_{i\alpha}(x)$ is a flavor doublet ($i=1, 2$ or $i=u, d$) with corresponding Pauli matrices τ_k ($k=1, 2, 3$) and color N_c-plet ($\alpha=1,\ldots,N_c$), as well as a two-component Dirac spinor. (The summation in (1) over flavor, color, and spinor indices is implied.) The quark number chemical potential μ in (1) is responsible for the nonzero baryonic density of quark matter, whereas the isospin chemical potential μ_I is taken into account in order to study asymmetric quark matter at nonzero isospin densities. (In this case, the densities of u and d quarks are different.) Evidently, the model (1) is a generalization of the original (1 + 1)-dimensional Gross-Neveu model [12] with a single quark to the case of two quark flavors and additional chemical potentials. As a result, we have for our model a more complicated chiral symmetry group. Indeed, at $\mu_I=0$ apart from the global color $SU(N_c)$ symmetry, the Lagrangian (1) is invariant under transformations of the chiral $SU_L(2)\times SU_R(2)$ group. However, at $\mu_I\neq 0$, this symmetry is reduced to $U_{I_3L}(1)\times U_{I_3R}(1)$, where $I_3=\tau_3/2$ is the third component of the isospin operator. (As usual, the subscripts L, R mean that the corresponding group acts only on the left-, right-handed spinors, respectively.) Evidently, this symmetry can also be presented as $U_{I_3}(1)\times U_{AI_3}(1)$, where $U_{I_3}(1)$, $U_{AI_3}(1)$ denote the isospin and the axial isospin subgroups, respectively. Quarks are transformed under these subgroups as $q\to\exp(i\alpha\tau_3)q$ and $q\to\exp(i\alpha\gamma^5\tau_3)q$, respectively.[2] Notice that Lagrangian (1) is parity invariant.

[2]Recall that $\exp(i\alpha\tau_3)=\cos\alpha+i\tau_3\sin\alpha$, $\exp(i\alpha\gamma^5\tau_3)=\cos\alpha+i\gamma^5\tau_3\sin\alpha$.

The linearized version of Lagrangian (1), which contains composite bosonic fields $\sigma(x)$ and $\pi_a(x)$ ($a = 1, 2, 3$), has the following form:

$$\tilde{\mathcal{L}} = \bar{q}[\gamma^\rho i\partial_\rho + \mu\gamma^0 + \nu\tau_3\gamma^0 - \sigma - i\gamma^5\pi_a\tau_a]q - \frac{N_c}{4G}[\sigma^2 + \pi_a^2], \quad (2)$$

where $\nu = \mu_I/2$. Evidently, the Lagrangian (2) is equivalent to (1), which simply follows from the use of the following constraint equations for the bosonic fields:

$$\sigma(x) = -2\frac{G}{N_c}(\bar{q}q); \qquad \pi_a(x) = -2\frac{G}{N_c}(\bar{q}i\gamma^5\tau_a q). \quad (3)$$

Furthermore, it is clear from (3) and footnote 2 that the bosonic fields transform under the isospin $U_{I_3}(1)$ and axial isospin $U_{AI_3}(1)$ subgroups in the following manner:

$$\begin{aligned}
&U_{I_3}(1)\colon\ \sigma \to \sigma; \qquad \pi_3 \to \pi_3;\\
&\pi_1 \to \cos(2\alpha)\pi_1 + \sin(2\alpha)\pi_2;\\
&\pi_2 \to \cos(2\alpha)\pi_2 - \sin(2\alpha)\pi_1, \qquad U_{AI_3}(1)\colon\ \pi_1 \to \pi_1;\\
&\pi_2 \to \pi_2; \qquad \sigma \to \cos(2\alpha)\sigma + \sin(2\alpha)\pi_3;\\
&\pi_3 \to \cos(2\alpha)\pi_3 - \sin(2\alpha)\sigma. \qquad (4)
\end{aligned}$$

Starting from Lagrangian (2), one obtains in the leading order of the large N_c-expansion (i.e. in the one-fermion loop approximation) the following path integral expression for the effective action $\mathcal{S}_{\rm eff}(\sigma, \pi_a)$ of the bosonic $\sigma(x)$ and $\pi_a(x)$ fields:

$$\exp(i\mathcal{S}_{\rm eff}(\sigma, \pi_a)) = N' \int[d\bar{q}][dq]\exp\Big(i\int\tilde{\mathcal{L}}d^2x\Big),$$

where

$$\mathcal{S}_{\rm eff}(\sigma, \pi_a) = -N_c\int d^2x\Big[\frac{\sigma^2 + \pi_a^2}{4G}\Big] + \tilde{\mathcal{S}}_{\rm eff}, \quad (5)$$

and N' is a normalization constant. The quark contribution to the effective action, i.e. the term $\tilde{\mathcal{S}}_{\rm eff}$ in (5), is given by

$$\exp(i\tilde{\mathcal{S}}_{\rm eff}) = N'\int[d\bar{q}][dq]\exp\Big(i\int\{\bar{q}[\gamma^\rho i\partial_\rho + \mu\gamma^0 + \nu\tau_3\gamma^0 - \sigma - i\gamma^5\pi_a\tau_a]q\}d^2x\Big). \quad (6)$$

The ground state expectation values $\langle\sigma(x)\rangle$ and $\langle\pi_a(x)\rangle$ of the composite bosonic fields are determined by the saddle point equations,

$$\frac{\delta\mathcal{S}_{\rm eff}}{\delta\sigma(x)} = 0, \qquad \frac{\delta\mathcal{S}_{\rm eff}}{\delta\pi_a(x)} = 0, \quad (7)$$

where $a = 1, 2, 3$. In vacuum, i.e. in the state corresponding to an empty space with zero particle density and zero values of the chemical potentials μ and μ_I, the quantities $\langle\sigma(x)\rangle$ and $\langle\pi_a(x)\rangle$ do not depend on space coordinates. However, in a dense medium, when $\mu \neq 0$, $\mu_I \neq 0$, the ground state expectation values of bosonic fields might have a nontrivial dependence on x. In particular, in this paper we will use the following ansatz:

$$\begin{aligned}
&\langle\sigma(x)\rangle = M\cos(2bx), \qquad \langle\pi_3(x)\rangle = M\sin(2bx),\\
&\langle\pi_1(x)\rangle = \Delta, \qquad \langle\pi_2(x)\rangle = 0,
\end{aligned} \quad (8)$$

where M, b and Δ are constant quantities. In fact, they are coordinates of the global minimum point of the thermodynamic potential (TDP) $\Omega(M, b, \Delta)$.[3] In the leading order of the large N_c-expansion, it is defined by the following expression:

$$\int d^2x\Omega(M, b, \Delta) = -\frac{1}{N_c}\mathcal{S}_{\rm eff}\{\sigma(x), \pi_a(x)\}|_{\sigma(x)=\langle\sigma(x)\rangle, \pi_a(x)=\langle\pi_a(x)\rangle}, \quad (9)$$

which gives

$$i\int d^2x\Omega(M,b,\Delta) = i\int d^2x\frac{M^2+\Delta^2}{4G} - \frac{1}{N_c}\ln\Big(\int[d\bar{q}][dq]\times\exp\Big(i\int d^2x\bar{q}\mathcal{D}q\Big)\Big), \quad (10)$$

where

$$\mathcal{D} = \gamma^\rho i\partial_\rho + \mu\gamma^0 + \nu\tau_3\gamma^0 - M\exp(2i\gamma^5\tau_3 bx) - i\gamma^5\tau_1\Delta. \quad (11)$$

To proceed, let us introduce the new quark fields, $q_w = \exp(i\gamma^5\tau_3 bx)q$ and $\bar{q}_w = \bar{q}\exp(i\gamma^5\tau_3 bx)$, such that

$$\bar{q}\mathcal{D}q = \bar{q}_w[\gamma^\rho i\partial_\rho + \mu\gamma^0 + (b+\nu)\tau_3\gamma^0 - M - i\gamma^5\tau_1\Delta]q_w \equiv \bar{q}_w D q_w, \quad (12)$$

where instead of the x-dependent Dirac operator (11) a new x-independent operator appears:

$$D = \gamma^\rho i\partial_\rho + \mu\gamma^0 + (b + \nu)\tau_3\gamma^0 - M - i\gamma^5\tau_1\Delta. \quad (13)$$

Since this transformation of quark fields does not change the path integral measure in (10),[4] expression (10) for the thermodynamic potential is easily transformed into the following one:

[3]Here and in the following we will use a rather conventional notation, "global" minimum, in the sense that among all our numerically found local minima the thermodynamical potential takes in their case the lowest value. This does not exclude the possibility that there exist other inhomogeneous condensates, different from (8), which lead to ground states with even lower values of the TDP.

[4]This nontrivial fact follows from the investigations by Fujikawa [34], who established that a chiral transformation of spinor fields changes the path integral measure only in the case when there is an interaction between spinor and gauge fields.

$$\Omega(M, b, \Delta) = \frac{M^2 + \Delta^2}{4G} + i\frac{\mathrm{Tr}_{sfx}\ln D}{N_c \int d^2x}$$
$$= \frac{M^2 + \Delta^2}{4G} + i\,\mathrm{Tr}_{sf}\int\frac{d^2p}{(2\pi)^2}\ln(\not p + \mu\gamma^0 + (b+\nu)\tau_3\gamma^0 - M - i\gamma^5\Delta\tau_1), \quad (14)$$

where the Tr-operation Tr_{sfx} stands for the trace in spinor- (s), flavor- (f) as well as two-dimensional coordinate- (x) spaces, respectively, and Tr_{sf} is the respective trace without x-space. Since the thermodynamic potential (14) is formally equal to the TDP (9) of paper [22] when one performs the replacement $\nu \to b + \nu$, one can further use the corresponding techniques and obtain

$$\Omega(M, b, \Delta) = \frac{M^2 + \Delta^2}{4G} + i\int\frac{d^2p}{(2\pi)^2}\ln\{[(p_0 + \mu)^2 - (E_\Delta^+)^2][(p_0 + \mu)^2 - (E_\Delta^-)^2]\}, \quad (15)$$

where

$$E_\Delta^\pm = \sqrt{(E^\pm)^2 + \Delta^2}, \quad E^\pm = E \pm (b+\nu), \quad E = \sqrt{p_1^2 + M^2}. \quad (16)$$

The argument of the ln(x)-function in (15) is proportional to the inverse quark propagator in the energy-momentum space representation. Hence, its zeros are the poles of the quark propagator. So, using (15), one can find the dispersion laws for quasiparticles, i.e. the momentum dependence of the quark (p_{0u}, p_{0d}) and antiquark ($p_{0\bar{u}}$, $p_{0\bar{d}}$) energies, in a medium (the full expression of the quark propagator matrix is presented in Appendix B of paper [23]):

$$p_{0u} = E_\Delta^- - \mu, \qquad p_{0d} = E_\Delta^+ - \mu,$$
$$p_{0\bar{u}} = -(E_\Delta^+ + \mu), \qquad p_{0\bar{d}} = -(E_\Delta^- + \mu). \quad (17)$$

It is clear that expression (15) is symmetric with respect to the transformations $\mu \to -\mu$ and $(b + \nu) \to -(b + \nu)$, respectively. Thus, without loss of generality, it is sufficient to adopt the restrictions $\mu \geq 0$ and $(b + \nu) \geq 0$. Under these conditions, upon integrating in (15) over p_0 (see Ref. [22] for similar integrals), one obtains for the TDP of the system at zero temperature the following expression:

$$\Omega(M, b, \Delta) = \frac{M^2 + \Delta^2}{4G} - \int_0^\infty\frac{dp_1}{\pi}\{E_\Delta^+ + E_\Delta^- + (\mu - E_\Delta^+)\theta(\mu - E_\Delta^+) + (\mu - E_\Delta^-)\theta(\mu - E_\Delta^-)\}, \quad (18)$$

where $\theta(x)$ is the Heaviside theta-function.

III. HOMOGENEOUS CHIRAL CONDENSATE, $b = 0$

A. The case with flavor symmetry, $\mu_I = 0$

First of all, let us consider the vacuum case, i.e. when $\mu = 0$, $\mu_I = 0$ and temperature is zero. Since in QCD parity is not broken in the vacuum, it is necessary in all QCD-motivated theories to adopt the same requirement. In the framework of our model this means that one should take $\Delta = 0$ if $\mu = 0$, $\mu_I = 0$. Assuming homogeneity of the chiral condensate ($b = 0$), we then obtain from (18) the following expression for the effective potential of the initial NJL_2 model in vacuum ($\Delta = 0$, $\mu = 0$, $\mu_I = 0$)[5]:

$$V_0(M) = \frac{M^2}{4G} - \frac{2}{\pi}\int_0^\infty dp_1\sqrt{p_1^2 + M^2}. \quad (19)$$

Formally, the effective potential (19) is a UV-divergent quantity. To renormalize $V_0(M)$, i.e. to obtain a finite expression for it, we first need to regularize the integral in the right-hand side of (19) by cutting off its integration region, $p_1 < \Lambda$. Second, we suppose that the bare coupling constant G in (19) depends on the cutoff parameter Λ ($G \equiv G(\Lambda)$) in such a way that in the limit $\Lambda \to \infty$ one obtains a finite expression. To construct the function $G(\Lambda)$, let us suppose that the stationarity equation $\partial V_0(M)/\partial M = 0$ has a nontrivial solution M_0. Then it is easy to obtain from this equation the following expression for the bare coupling constant $G(\Lambda)$:

$$\frac{1}{2G(\Lambda)} = \frac{2}{\pi}\int_0^\Lambda dp_1\frac{1}{\sqrt{M_0^2 + p_1^2}} = \frac{2}{\pi}\ln\left(\frac{\Lambda + \sqrt{M_0^2 + \Lambda^2}}{M_0}\right). \quad (20)$$

Now, using (20) in the regularized expression (19) and adding an unessential constant Λ^2/π, one can find at $\Lambda \to \infty$:

$$V_0(M) \equiv \lim_{\Lambda\to\infty}\left\{\frac{M^2}{4G(\Lambda)} - \frac{2}{\pi}\int_0^\Lambda dp_1\sqrt{p_1^2 + M^2} + \frac{\Lambda^2}{\pi}\right\} = \frac{M^2}{2\pi}\left[\ln\left(\frac{M^2}{M_0^2}\right) - 1\right]. \quad (21)$$

Since M_0 might be considered as a free model parameter, it follows from (20) and (21) that the renormalization procedure of the NJL_2 model is accompanied by the dimensional transmutation phenomenon. Indeed, in the initial unrenormalized expression (19) for $V_0(M)$ the dimensionless coupling constant G is present, whereas after renormalization the effective potential (21) is characterized by a dimensional free model parameter M_0. Moreover, as is clear from (21), the global minimum point of the effective potential

[5]In vacuum, the thermodynamic potential is usually called effective potential.

$V_0(M)$ lies just at the point $M = M_0$, so in vacuum the chiral $SU_L(2) \times SU_R(2)$ symmetry of the NJL$_2$ model (1) is always spontaneously broken and the quantity M_0 might be treated as dynamical quark mass (in vacuum).

Detailed information about the phase structure of the NJL$_2$ model (1) at $\mu \neq 0$, $\mu_I = 0$ can be found, e.g., in [14]. So, at $\mu > M_0/\sqrt{2}$ and $\mu_I = 0$ there is a massless chirally symmetric phase with nonzero baryon density. However, at $\mu < M_0/\sqrt{2}$ and $\mu_I = 0$, chiral symmetry is spontaneously broken down and quarks acquire a mass M_0. In this phase baryon density is equal to zero.

B. Phase structure in the general case: $\mu \neq 0$, $\mu_I \neq 0$

To find the phase portrait of the NJL$_2$ model (1) in the case of a homogeneous chiral condensate but for arbitrary values of chemical potentials and at zero temperature, one should start from the expression (18) with $b = 0$. (Note that, at $\mu_I \neq 0$, the condensation of charged pions might occur, so we need to take into account a nonzero value of Δ.) Obviously, this expression is again UV-divergent, so first of all it is necessary to regularize it. Using, as the most simple regularization, a Λ-cutoff in the one-dimensional momentum space, we have:

$$\Omega_{\rm reg}(M, b=0, \Delta) = \frac{M^2 + \Delta^2}{4G} - \int_0^\Lambda \frac{dp_1}{\pi}\{\mathcal{E}_\Delta^+ + \mathcal{E}_\Delta^-\} - \int_0^\infty \frac{dp_1}{\pi}\{(\mu - \mathcal{E}_\Delta^+)\theta(\mu - \mathcal{E}_\Delta^+) + (\mu - \mathcal{E}_\Delta^-)\theta(\mu - \mathcal{E}_\Delta^-)\}, \quad (22)$$

where $\mathcal{E}_\Delta^\pm$ denotes the quantity $E_\Delta^\pm$ (16) at $b = 0$. Because of the presence of θ-functions, the second integral in (22) has a finite integration region, i.e. it is a proper integral that does not need to be regularized. To obtain a finite (renormalized) expression $\Omega(M, \Delta)$ for the thermodynamic potential, one should again perform in (22) the replacement $G \to G(\Lambda)$, the last quantity being given in (20), and then let Λ tend to infinity (compare with (21)), i.e.

$$\Omega(M, \Delta) = \lim_{\Lambda\to\infty}\left\{\Omega_{\rm reg}(M, b=0, \Delta)|_{G\to G(\Lambda)} + \frac{\Lambda^2}{\pi}\right\}. \quad (23)$$

Using the definition of the effective potential in vacuum [see (21)], it is easy to obtain the following renormalization invariant expression of the TDP (23):

$$\Omega(M, \Delta) = V_0(\sqrt{M^2 + \Delta^2}) - \int_0^\infty \frac{dp_1}{\pi}\{\mathcal{E}_\Delta^+ + \mathcal{E}_\Delta^- - 2\sqrt{p_1^2 + M^2 + \Delta^2}\} - \int_0^\infty \frac{dp_1}{\pi}\{(\mu - \mathcal{E}_\Delta^+)\theta(\mu - \mathcal{E}_\Delta^+) + (\mu - \mathcal{E}_\Delta^-)\theta(\mu - \mathcal{E}_\Delta^-)\}, \quad (24)$$

where the function $V_0(x)$ is defined in (21). Moreover, the second integral in (24) is proper [see also the corresponding remark just after (22)], whereas the first integral is convergent and defined as

$$\int_0^\infty dp_1[\mathcal{E}_\Delta^+ + \mathcal{E}_\Delta^- - 2\sqrt{p_1^2 + M^2 + \Delta^2}] = \lim_{\Lambda\to\infty}\left\{\int_0^\Lambda dp_1[\mathcal{E}_\Delta^+ + \mathcal{E}_\Delta^- - 2\sqrt{p_1^2 + M^2 + \Delta^2}]\right\}. \quad (25)$$

Thus, in the case of a homogeneous chiral condensate, the TDP is given by (23) and (24) and the corresponding phase structure, following from it, is depicted in Fig. 1. (For a more detailed investigation of this TDP, see paper [24].) In the figure, the phases denoted by 1 and 2 correspond to the global minimum point (GMP) of the form ($M = 0$, $\Delta = 0$) and ($M \neq 0$, $\Delta = 0$), correspondingly. In the pion condensed phase (PC), the GMP of the TDP (24) has the form ($M = 0$, $\Delta = M_0$), i.e. in this phase the isospin symmetry $U_{I_3}(1)$ is broken spontaneously.[6] It is easy to see that throughout the PC phase the quark number density is equal to zero, whereas the isospin density $n_I = -\partial\Omega/\partial\mu_I$ is equal to ν/π.

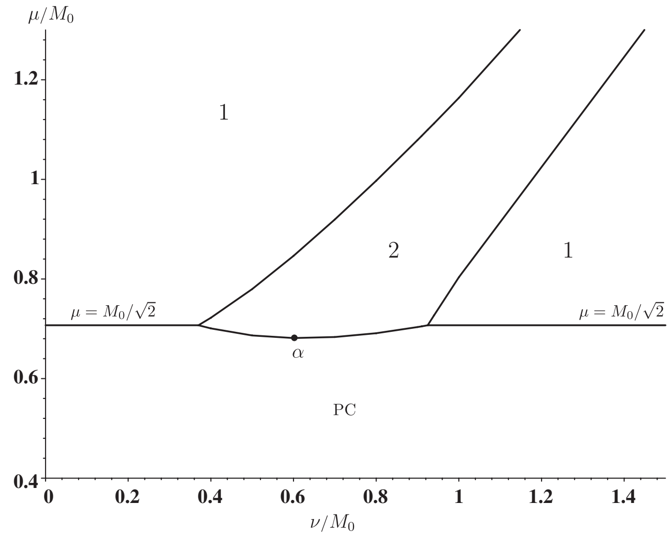

FIG. 1. The (μ, ν) phase portrait of the model considered at $T = 0$ and $\nu > 0$ in the case of spatially homogeneous condensates. Here $\nu = \frac{\mu_I}{2}$, and M_0 is the quark mass in the vacuum. Number 1 denotes the symmetric phase with massless quarks, number 2 denotes the normal quark matter phase with massive quarks, and PC denotes the charged pion condensed phase. The point α is the lowest point of the phase 2: $\mu_\alpha \approx 0.68M_0$, $\nu_\alpha \approx 0.6M_0$.

[6]Note that our numerical investigations show that the TDP (24) has no local minima of the form ($M \neq 0$, $\Delta \neq 0$), i.e. simultaneous dynamical quark mass generation and charged pion condensation are incompatible in the framework of the NJL$_2$ model (1) at $b = 0$. The same is valid for the simple two-flavored NJL$_4$ model in the mean-field approximation [10]. However, it is not excluded that there might be realized a mixed phase with both nonzero gaps, $M \neq 0$ and $\Delta \neq 0$, in models with a more complicated four-fermion structure.

IV. INHOMOGENEOUS CHIRAL CONDENSATE, $b \neq 0$

To obtain the phase portrait of the initial NJL_2 model in this case (temperature T is zero), let us start from the most general expression for the TDP (18). As previously, let us first use the most simple momentum cutoff regularization of this quantity,

$$\Omega_{\rm reg}(M,b,\Delta) = \frac{M^2+\Delta^2}{4G} - \int_0^\Lambda \frac{dp_1}{\pi}\{E_\Delta^+ + E_\Delta^-\} - \int_0^\infty \frac{dp_1}{\pi}\{(\mu - E_\Delta^+)\theta(\mu - E_\Delta^+) + (\mu - E_\Delta^-)\theta(\mu - E_\Delta^-)\}, \quad (26)$$

where the expressions for $E_\Delta^\pm$ are presented in (16). The corresponding renormalized expression for the TDP is again defined by [compare with (23)]

$$\Omega(M,b,\Delta) = \lim_{\Lambda\to\infty}\left\{\Omega_{\rm reg}(M,b,\Delta)|_{G\to G(\Lambda)} + \frac{\Lambda^2}{\pi}\right\}, \quad (27)$$

where $G(\Lambda)$ is given in (20), and reads

$$\Omega(M,b,\Delta) = V_0(\sqrt{M^2+\Delta^2}) - \lim_{\Lambda\to\infty}\left\{\int_0^\Lambda \frac{dp_1}{\pi}\left[E_\Delta^+ + E_\Delta^- - 2\sqrt{p_1^2+M^2+\Delta^2}\right]\right\} - \int_0^\infty \frac{dp_1}{\pi}\{(\mu - E_\Delta^+)\theta(\mu - E_\Delta^+) + (\mu - E_\Delta^-)\theta(\mu - E_\Delta^-)\}. \quad (28)$$

(Evidently, at $b = 0$, this expression coincides with the TDP $\Omega(M,\Delta)$ (24).) The global minimum point of the function $\Omega(M,b,\Delta)$ (28) vs variables M, b and Δ should render the phase structure of the model. However, two circumstances prevent us from considering this quantity as a genuine physical thermodynamic potential of the system. The first is that the function (28) is not bounded from below with respect to the variable b. Second, it is intuitively clear that at $M = 0$ the genuine thermodynamic potential should not depend on the variable b, because no observable quantity may depend on a wave vector if the amplitude of the corresponding oscillations (wave) is zero. However, the TDP defined by (28) at $M = 0$ (see also in [24]),

$$\Omega(M=0,b,\Delta) = V_0(\Delta) - \frac{(b+\nu)^2}{\pi} + \frac{\theta(\mu-\Delta)}{\pi}\times\left[\Delta^2\ln\left(\frac{\mu+\sqrt{\mu^2-\Delta^2}}{\Delta}\right) - \mu\sqrt{\mu^2-\Delta^2}\right], \quad (29)$$

retains an unphysical dependence on b. Clearly, the two above-mentioned unphysical properties of the TDP (28) are due to the term $-\frac{(b+\nu)^2}{\pi}$ in (29). Hence, the subtraction of this term from the TDP (28) brings us to the quantity, which might serve as a physically acceptable thermodynamic potential of the system,

$$\Omega^{\rm phys}(M,b,\Delta) = \Omega(M,b,\Delta) + \frac{(b+\nu)^2}{\pi} - \frac{\nu^2}{\pi}. \quad (30)$$

(We also add in the expression (30) a b-independent term, $-\nu^2/\pi$, in order to reproduce at $b = 0$ the TDP (24), corresponding to a spatially homogeneous chiral condensate.) The reason the expression for the TDP (30) does not follow straightforwardly from the unrenormalized TDP expression (18) lies in the usage of the *symmetric momentum cutoff regularized* TDP (26). This means that, for each energy $E_\Delta^\pm$, the integration in the first (regularized) integral of (26) is performed over the same momentum interval $0 < p_1 < \Lambda$. Correspondingly, in this case there is an asymmetry in values of energies $E_\Delta^\pm$, which contribute to $\Omega_{\rm reg}(M,b,\Delta)$. Indeed, if $p_1 < \Lambda$, then

$$E_\Delta^\pm < \sqrt{(\sqrt{\Lambda^2+M^2} \pm (b+\nu))^2 + \Delta^2},$$

i.e. for different quasiparticles there are allowed different regions of their energy values. However, as discussed in the recent papers [28,29], a more adequate regularization scheme in the case of spatially inhomogeneous phases is that one where there is an energy constraint which is the same for all quasiparticles. So, dealing with spatial inhomogeneity, one can use, e.g., the Schwinger proper-time regularization, dimensional regularization, etc. In particular, in the recent paper [31], the *symmetric energy cutoff regularization* scheme was proposed in considering the behavior of chiral density waves in the presence of an external magnetic field in the framework of a four-dimensional Nambu–Jona-Lasinio model. There, for each quasiparticle the same (finite) interval of their energy values was allowed to contribute to the regularized thermodynamic potential. As a result, a physically relevant renormalized TDP without the above-mentioned shortcomings was obtained.

In this paper the slightly modified energy cutoff regularization scheme of [31] is adopted. Namely, we require that only energies with momenta p_1, constrained by the relations $E_\Delta^\pm(M=0,\Delta=0) = p_1 \pm (b+\nu) < \Lambda$, contribute to the regularized thermodynamic potential. This means that the term with energy E_Δ^+ (E_Δ^-) should be integrated in the regularized expression for TDP over the interval $0 < p_1 < \Lambda - (b+\nu)$ ($0 < p_1 < \Lambda + (b+\nu)$). (A similar regularization was used in studying the CDW phase in a two-dimensional NJL model without isospin chemical potential [17].) Consequently, we have the following regularized expression for the TDP (18):

$$\tilde{\Omega}_{\rm reg}(M,b,\Delta)=\frac{M^2+\Delta^2}{4G}-\frac{1}{\pi}\int_0^{\Lambda-\tilde{\nu}}dp_1E_\Delta^+ -\frac{1}{\pi}\int_0^{\Lambda+\tilde{\nu}}dp_1E_\Delta^- -\int_0^\infty\frac{dp_1}{\pi}\{(\mu-E_\Delta^+)\times\theta(\mu-E_\Delta^+)+(\mu-E_\Delta^-)\theta(\mu-E_\Delta^-)\}, \quad (31)$$

where $\tilde{\nu}=(b+\nu)$. Replacing in this formula G by $G(\Lambda)$ from (20) and adding an unessential constant $(\Lambda^2-\nu^2)/\pi$, we obtain a physically "improved" renormalized expression $\tilde{\Omega}(M,b,\Delta)$ for the TDP (18) when $\Lambda\to\infty$, which differs from the expression $\Omega(M,b,\Delta)$ in (28). Comparing (26) and (31) one can easily find that

$$\tilde{\Omega}(M,b,\Delta)-\Omega(M,b,\Delta) =\lim_{\Lambda\to\infty}\left\{\frac{1}{\pi}\int_{\Lambda-\tilde{\nu}}^{\Lambda}dp_1E_\Delta^+ -\frac{1}{\pi}\int_{\Lambda}^{\Lambda+\tilde{\nu}}dp_1E_\Delta^-\right\} =\frac{(b+\nu)^2}{\pi}-\frac{\nu^2}{\pi}. \quad (32)$$

(To obtain the last expression in (32) one should take into account that at $\Lambda\to\infty$ the p_1-values in both integrals are much greater than M, Δ, b, μ, μ_I. In this case it is possible to expand the quantities $E_\Delta^\pm$ into power series of p_1 and then to integrate each term.) Comparing (30) and (32), we see that $\tilde{\Omega}(M,b,\Delta)=\Omega^{\rm phys}(M,b,\Delta)$, i.e. there exists a regularization scheme,[7] which in the case of the inhomogeneous chiral condensate (8) brings us to a physically acceptable TDP $\Omega^{\rm phys}(M,b,\Delta)$ (30). Notice also that if $b=0$ then $\Omega(M,b,\Delta)$ is equal to $\Omega^{\rm phys}(M,b,\Delta)$. Hence, in the case of homogeneous chiral condensates the two above-considered regularization schemes are equivalent. In contrast, in the inhomogeneous case the thermodynamic potentials $\Omega(M,b,\Delta)$ and $\Omega^{\rm phys}(M,b,\Delta)$ differ by terms, containing the dynamical quantity b. As a result, the regularizations are not equivalent. However, since the symmetric momentum cutoff regularization is easier to handle, it is possible to perform all calculations in the framework of that regularization scheme and then simply correct the obtained TDP $\Omega(M,b,\Delta)$ by the terms $\frac{(b+\nu)^2}{\pi}-\frac{\nu^2}{\pi}$ [see (30)], instead of using from the beginning one of the physically acceptable regularizations bringing us directly to the TDP $\Omega^{\rm phys}(M,b,\Delta)$.

[7]Moreover, we expect that any regularization scheme, in which there is a constraint on the energy values common for all quasiparticles, should provide us with TDP $\Omega^{\rm phys}(M,b,\Delta)$ (30). Among these regularizations are dimensional and analytical ones, Pauli-Villars and Schwinger prope-time regularizations, as well as the above-mentioned symmetric energy cutoff regularization [31]. In particular, the proper-time regularization is often used in studying inhomogeneous phases in the framework of NJL models [28,29] and does not lead to any unphysical effects, etc.

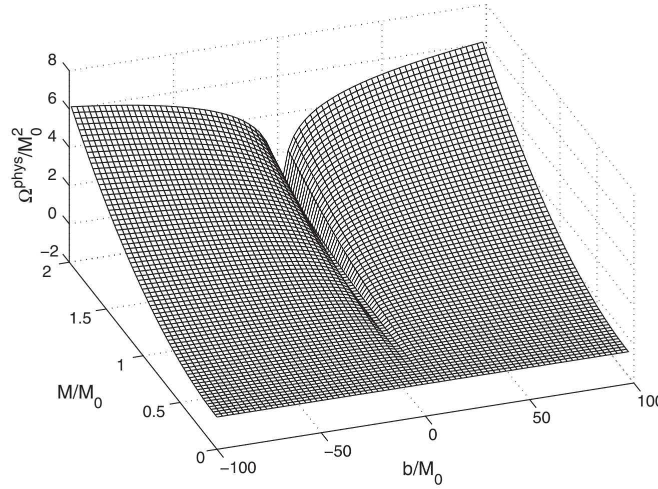

FIG. 2. The plot of $\Omega^{\rm phys}$ (30) vs M, b at $\mu_I=0$, $\mu=M_0$, $\Delta=0$.

To illustrate the fact that the TDP $\Omega^{\rm phys}(M,b,\Delta)$ is bounded from below as a function of the variable b, we plot in Fig. 2 this thermodynamic potential vs M, b in the particular case $\mu_I=0$, $\Delta=0$, $\mu=M_0$.

A. Particular case: $\mu_I=0$, $\mu\neq 0$

Recall that the CDW inhomogeneous phase was established earlier in the NJL$_2$ model with $U_L(1)\times U_R(1)$ chiral symmetry for all $\mu>0$ at rather low temperatures [17,18]. In contrast, in this paper we are going to study chiral density waves in the NJL$_2$ model with a continuous chiral $SU_L(2)\times SU_R(2)$ symmetry. In the present section we consider the case of $T=0$. It is well-known that at $\mu_I=0$ the charged pion condensation phenomenon is forbidden (see, e.g., [24]), so without loss of generality one may suppose that $\Delta=0$ in (28). Then the TDP $\Omega(M,b,\Delta=0)$ can be easily evaluated analytically (see [24]) and the physical thermodynamic potential $\Omega^{\rm phys}(M,b)\equiv\Omega^{\rm phys}(M,b,\Delta=0)$ (30) looks like

$$\Omega^{\rm phys}(M,b)=V_0(M)+\frac{\theta(\mu+b-M)}{2\pi}\times\left[M^2\ln\left(\frac{\mu+b+\sqrt{(\mu+b)^2-M^2}}{M}\right)-(\mu+b)\sqrt{(\mu+b)^2-M^2}\right]+\frac{\theta(|\mu-b|-M)}{2\pi}\times\left[M^2\ln\left(\frac{|\mu-b|+\sqrt{(\mu-b)^2-M^2}}{M}\right)-|\mu-b|\sqrt{(\mu-b)^2-M^2}\right]+\frac{b^2}{\pi}. \quad (33)$$

Recall that in (33) the constraints $\mu\geq 0$, $b\geq 0$, $M\geq 0$ are supposed. The phase structure of the model in this particular case is defined by the properties of the global minimum point (GMP) of the TDP (33), which certainly depend on the values of μ. The stationarity (gap) equations of this TDP, i.e. the equations $\partial\Omega^{\rm phys}(M,b)/\partial M = 0$ and $\partial\Omega^{\rm phys}(M,b)/\partial b = 0$, read:

$$M\left\{\ln\left(\frac{M^2}{M_0^2}\right)+\theta(\mu+b-M)\ln\left(\frac{\mu+b+\sqrt{(\mu+b)^2-M^2}}{M}\right)+\theta(|\mu-b|-M)\ln\left(\frac{|\mu-b|+\sqrt{(\mu-b)^2-M^2}}{M}\right)\right\}=0, \tag{34}$$

$$2b=\theta(\mu+b-M)\sqrt{(\mu+b)^2-M^2}+\operatorname{sign}(b-\mu)\theta(|b-\mu|-M)\sqrt{(b-\mu)^2-M^2}. \tag{35}$$

Numerical investigations of the TDP (33) and of the gap equations (34) and (35) show that in the NJL_2 model with chiral $SU_L(2)\times SU_R(2)$ symmetry the inhomogeneous CDW phase is realized only at $\mu > \mu_c \approx 0.68M_0$. In contrast, at $T = 0$, in the $(1+1)$-dimensional $U_L(1)\times U_R(1)$ chirally symmetric model, the CDW phase appears at arbitrary nonzero values of μ [17,18]. Note that the critical value μ_c is equal to μ_α which corresponds to the lowest point of the homogeneous phase 2 (see Fig. 1). Below the critical chemical potential, i.e. at $\mu < \mu_c$, the usual homogeneous phase is arranged, where chiral symmetry is broken down to the diagonal $SU(2)$ subgroup. The behavior of the chiral density wave amplitude $M(\mu)$ and its wave vector $b_0(\mu)$, which are the coordinates of the global minimum point of the TDP (33), is shown in Fig. 3 for $\mu_I = 0$. It follows from this figure that at the critical point μ_c a first order phase transition takes place, since here the order parameter M changes its value by a jump. Since in the CDW phase the relation $M(\mu) < b_0(\mu) < \mu$ is valid, it is clear from the dispersion laws (17) at $\Delta = 0$ that u-quarks are gapless excitations of this phase. It means that for each $\mu > \mu_c$ there exists a momentum $p_1(\mu)$ at which the quasiparticle energy p_{0u} is equal to zero, i.e. there is no energy cost to create u-quarks in the system. In contrast, for the energy of d-quarks we have throughout the CDW phase the relation $p_{0d} > p_{0\,\rm min} = M(\mu) + b_0(\mu) - \mu \approx M(\mu)$, i.e. there is a gap in the energy spectrum of d-quarks which are called, for this reason, gapped excitations of the CDW phase. There is one more peculiarity of the CDW phase. Indeed, as is easily seen from (33), at $\nu = 0$ the effective quark number chemical potential of u-quarks is equal to $\mu + b$, whereas for d-quarks it is $\mu - b$. Hence, there is a splitting of Fermi surfaces

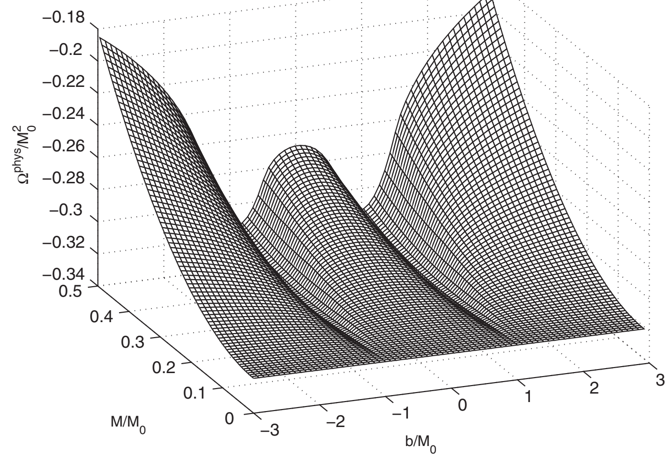

FIG. 4. The plot of $\Omega^{\rm phys}$ (33) vs M, b at $\mu = M_0$.

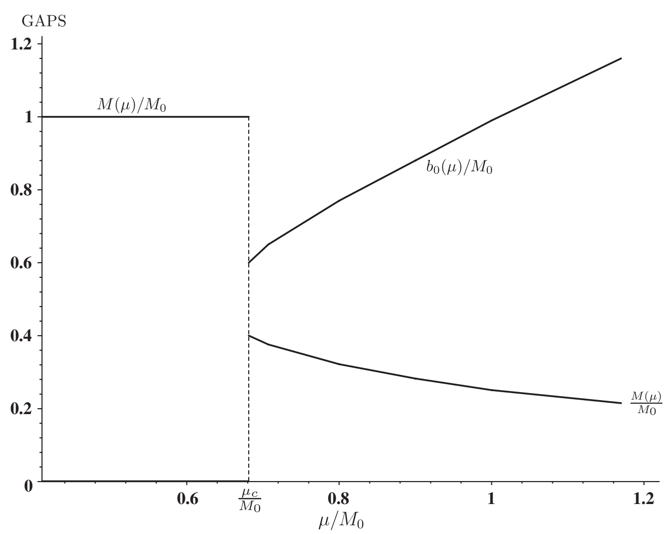

FIG. 3. The CDW amplitude $M(\mu)$ and its wave vector $b_0(\mu)$ as functions of μ in the case of zero isospin chemical potential. Here, $\mu_c = \mu_\alpha \approx 0.68M_0$.

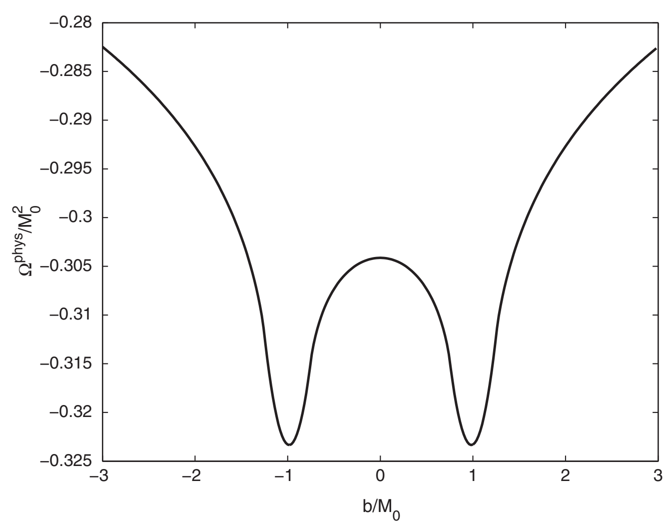

FIG. 5. Section of the plot of $\Omega^{\rm phys}$ (33) vs b at $\mu = M_0$ along the axis b passing through the point of minimum.

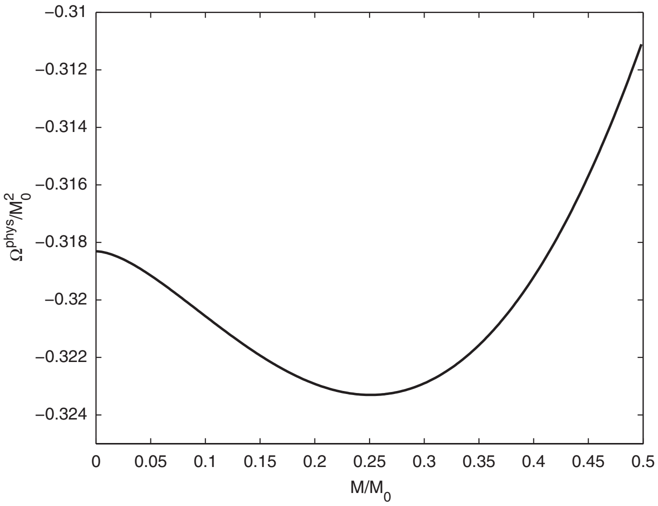

FIG. 6. Section of the plot of $\Omega^{\rm phys}$ (33) vs M at $\mu = M_0$ along the axis M passing through the point of minimum.

of up/down quarks by $2b_0(\mu)$ in the CDW phase even at zero μ_I.

The fact that at $\mu = M_0$ the TDP (33) has a nontrivial minimum at the point ($M \approx 0.25M_0$, $b \approx 0.99M_0$) is well-supported by Figs. 4–6, where the plot of the function $\Omega^{\rm phys}(M, b)$ of M and b and its sections along axes b and M are presented. (Note that in order to draw the figures we continue the function (33) symmetrically onto the negative semiaxis b.)

The influence of nonzero temperature on the formation of CDWs in the case $\mu_I = 0$ is considered, in particular, in Sec. V. (See Fig. 8.)

B. General case: $\mu_I \neq 0$, $\mu \neq 0$

It is clear that to find the complete phase portrait of the model in terms of the external chemical potential parameters $\nu \equiv \mu_I/2$ and μ (at $T = 0$), one should investigate the global minimum point (GMP) of the physical TDP $\Omega^{\rm phys}(M, b, \Delta)$ (30) vs the dynamical variables M, b, Δ.[8] However, in the case under consideration, the problem is simplified due to the effective reduction of external parameters. Indeed, the structure of $\Omega^{\rm phys}(M, b, \Delta)$ is such that it can be considered as a function of three dynamical variables M, Δ, $\tilde{\nu} \equiv b + \nu$, and only one external parameter μ, i.e. $\Omega^{\rm phys}(M, b, \Delta) \equiv F(M, \Delta, \tilde{\nu}; \mu)$. So, the searching for the GMP of this function consists effectively of two stages. First, one can find the extremum of this function over M and Δ (taking into account the results of Sec. III B) and then, as was done in Sec. IVA, one minimizes the obtained expression over the variable $\tilde{\nu}$. Properties of the found GMP supply the following phase structure.

If $\mu > \mu_\alpha = \mu_c \approx 0.68M_0$, then for arbitrary values of ν we have found phases with spatially inhomogeneous condensates, which are realized at least in the form of chiral density waves or chiral spirals. The gap Δ is equal to zero for these phases. The amplitude M of these CDWs depends only on μ and is equal to the quantity $M(\mu)$ (see Fig. 3). However, the chiral density wave vector b depends on both μ and ν, namely,

$$b = b_0(\mu) - \nu, \tag{36}$$

where the quantity $b_0(\mu)$ is also presented in Fig. 3. In the (ν, μ)-plane (see Fig. 7), we divide this region into two CDW phases. In the CDW$_1$ region we have the wave vector $b > 0$, i.e. here we have a clockwise twisted chiral spiral. In contrast, in the CDW$_2$ region, one obtains for chiral density waves the counterclockwise twisted chiral spirals, since here $b < 0$. For all points of the line L of this figure, which is defined by the relation $L = \{(\nu, \mu)\colon \nu = b_0(\mu)\}$, the wave vector b is equal to zero. So, the points of the curve L correspond to the homogeneous phase, where only chiral symmetry is spontaneously broken down and the dynamical quark mass is equal to the quantity $M(\mu)$ from Fig. 3. (Hence, on the line L, the spatial translational invariance of the system remains intact.) Note that the phase L is nothing else than the residue of the homogeneous phase 2 of Fig. 1 if the spatial inhomogeneity of chiral condensates is taken into account. To underline this fact, we use in Fig. 7 the notation μ_α, which corresponds to the minimum point α of the homogeneous phase 2 of Fig. 1, for the critical curve between the CDW and charged pion condensation (PC) phases. However, μ_α coincides with the critical value μ_c of the case $\mu_I = 0$ (see Sec. IVA).

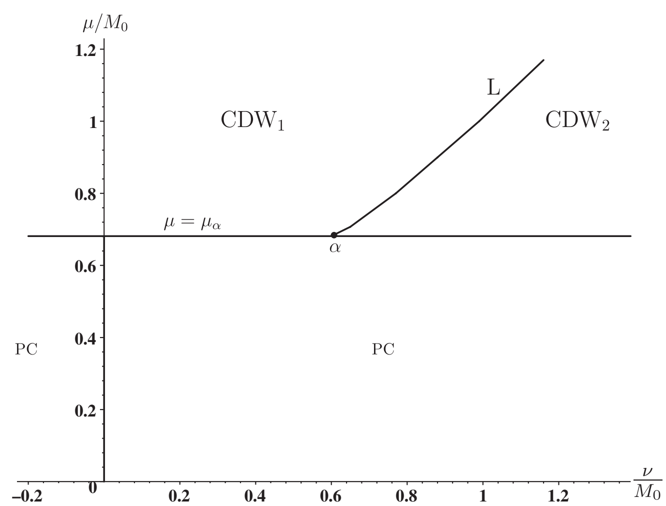

FIG. 7. The (ν, μ) phase portrait of the model at $T = 0$ when a spatial CDW inhomogeneity is taken into account. In the CDW$_1$ (CDW$_2$) phase, $b > 0$ ($b < 0$). The curve L, on which $b = 0$, corresponds to the homogeneous chiral symmetry broken phase. The same is true for the interval $0 < \mu < \mu_\alpha$ of the μ-axis, where $\mu_\alpha = \mu_c \approx 0.68M_0$.

[8]As in the case with $b = 0$, in the inhomogeneous case we did not find local minima of the TDP (30) of the form ($M \neq 0$, $\Delta \neq 0$).

As in the particular case with $\mu_I = 0$ (see the previous section), u-quarks are gapless excitations and d-quarks are gapped ones of the $CDW_{1,2}$ phases at $\mu_I \neq 0$. The same is true for the homogeneous phase L.

Below the line $\mu = \mu_\alpha$ of Fig. 7 the homogeneous PC phase is arranged, since for all points of this region the GMP of the TDP (30) has the form $M = 0$, $\Delta = M_0$, $b = 0$. In this phase the isospin $U_{I_3}(1)$ symmetry of the model is broken spontaneously. The exception is the interval $0 < \mu < \mu_\alpha$ of the μ-axis, where chiral symmetry is broken down and quarks have the mass M_0.

Note that, for both the case of spatially homogeneous and the case of inhomogeneous chiral condensate, the isospin density n_I in the PC phase is equal to ν/π. Starting from the $\Omega^{\rm phys}(M, b, \Delta)$ (30), it is possible to find the expression of this TDP in the $CDW_{1,2}$ phases [it is simply the expression (33) shifted by $(-\nu^2/\pi)$, in which M, b should be replaced by $M(\mu)$, $b_0(\mu)$, correspondingly] and then to calculate their isospin density $n_I = -\partial\Omega^{\rm phys}/\partial\mu_I$. It turns out that in the CDW phases the isospin density is the same as in the PC phase, i.e. $n_I = \nu/\pi$. Hence, as is easily seen from (36), at fixed values of μ the wave vector of chiral spirals is tightly (linearly) connected with isospin density. In contrast, in the $U_L(1) \times U_R(1)$ symmetric NJL_2 model without isospin chemical potential μ_I, the wave vector b shifts effectively the quark number chemical potential μ [17,18]. For this reason, the quark number density n_q is equal to μ/π in the CDW phase of this model. Moreover, the wave vector b in this phase is proportional to n_q.

V. CDW PHASES AT NONZERO TEMPERATURES

In the case of spatially homogeneous condensates the influence of nonzero temperature on the phase structure of the $SU_L(2) \times SU_R(2)$ symmetric NJL_2 model (1)with two chemical potentials μ and $\nu \equiv \mu_I/2$ was considered in [24]. Now let us study the influence of temperature T on the phase structure of this model in the case of an inhomogeneous chiral condensate of the form (8). In this case, to get the corresponding (unrenormalized) thermodynamic potential $\Omega_T(M, b, \Delta)$, one can simply start from the expression for the TDP at zero temperature (15) and perform the following standard replacements:

$$\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}(\cdots) \to iT\sum_{n=-\infty}^{\infty}(\cdots),\qquad p_0 \to p_{0n} \equiv i\omega_n \equiv i\pi T(2n+1),\qquad n = 0, \pm 1, \pm 2, \ldots, \tag{37}$$

i.e. the p_0-integration should be replaced by the summation over an infinite set of Matsubara frequencies ω_n. Summing over Matsubara frequencies in the obtained expression (the corresponding technique is presented, e.g. in [35]), one can find for the TDP:

$$\Omega_T(M,b,\Delta) = \frac{M^2+\Delta^2}{4G} - \int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\{E_\Delta^+ + E_\Delta^- + T\ln[1+e^{-\beta(E_\Delta^+-\mu)}] + T\ln[1+e^{-\beta(E_\Delta^++\mu)}] + T\ln[1+e^{-\beta(E_\Delta^--\mu)}] + T\ln[1+e^{-\beta(E_\Delta^-+\mu)}]\}, \tag{38}$$

where $\beta = 1/T$ and $E_\Delta^\pm$ are given in (16). Clearly, only the first two terms in the braces of this expression (which are the same as in the zero temperature case) are responsible for an ultraviolet divergency of the whole TDP (38). So, regularizing the TDP (38) in the way it was done in (31) for zero temperature TDP and then replacing $G \to G(\Lambda)$ [see formula (20)], we can obtain in the limit $\Lambda \to \infty$ a finite expression denoted as $\Omega_T^{\rm phys}(M, b, \Delta)$. It is an evident generalization of the TDP $\Omega^{\rm phys}(M, b, \Delta)$ (30) to the case of nonzero temperature. Numerical investigations show that all possible local minima of the obtained TDP $\Omega_T^{\rm phys}(M, b, \Delta)$ are located in the planes $M = 0$ or $\Delta = 0$. So it is sufficient to deal with corresponding restrictions of the TDP on these planes, i.e. with the following functions:

$$\Omega_T^{\rm phys}(M=0,b,\Delta) = V_0(\Delta) - \frac{2T}{\pi}\int_0^\infty dp_1\ln\{[1+e^{-\beta(\mathcal{E}-\mu)}][1+e^{-\beta(\mathcal{E}+\mu)}]\}, \tag{39}$$

$$\Omega_T^{\rm phys}(M,b,\Delta=0) = V_0(M) - \frac{(\nu+b)^2}{\pi} - \frac{T}{\pi}\int_0^\infty dp_1\ln\{[1+e^{-\beta(E+\nu+b-\mu)}][1+e^{-\beta(E+\nu+b+\mu)}]\} - \frac{T}{\pi}\int_0^\infty dp_1\ln\{[1+e^{-\beta(E-\nu-b-\mu)}][1+e^{-\beta(E-\nu-b+\mu)}]\}, \tag{40}$$

where the effective potential $V_0(x)$ is given in (21), $E = \sqrt{p_1^2+M^2}$, and $\mathcal{E} = \sqrt{p_1^2+\Delta^2}$. Comparing the global minima of the functions (39) and (40), it is possible to establish the global minimum point of the renormalized TDP $\Omega_T^{\rm phys}(M, b, \Delta)$. Then, the dependence of the global minimum point vs T, μ, ν defines the phase structure of the model.

Using this prescription in our numerical investigations of the TDPs (39) and (40), we have found the two (μ, T)-phase portraits of the initial NJL$_2$ model (1) depicted in Figs. 8 and 9 for qualitatively different fixed values of the isospin chemical potentials, $0 \le \mu_I < 2\nu_\alpha$ and $2\nu_\alpha < \mu_I$, respectively. ($\nu_\alpha \approx 0.6M_0$ is the ν-coordinate of the point α of Fig. 1.) Note, there is a *first order phase transition* on the boundaries between CDW$_{1,2}$ and homogeneous PC or chiral symmetry breaking phases of these figures. However, other boundaries of the phases of Figs. 8 and 9 correspond

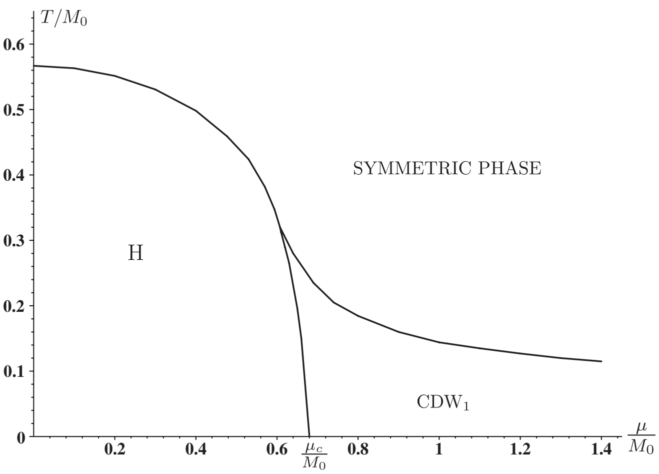

FIG. 8. The (μ, T) phase portrait of the model at fixed ν, where $0 \le \nu < \nu_\alpha \approx 0.6M_0$. There, at $\nu = 0$, H denotes the homogeneous chiral symmetry breaking phase with $M = M_0$, $b = 0$, $\Delta = 0$. At $0 < \nu < \nu_\alpha$ H denotes the homogeneous charged pion condensation phase (PC) with $M = 0$, $b = 0$, $\Delta = M_0$. $\mu_c = \mu_\alpha \approx 0.68M_0$. In the symmetric phase $M = 0$, $b = 0$, $\Delta = 0$. CDW$_1$ denotes an inhomogeneous chiral density wave phase with $b > 0$. All critical curves do not depend on ν.

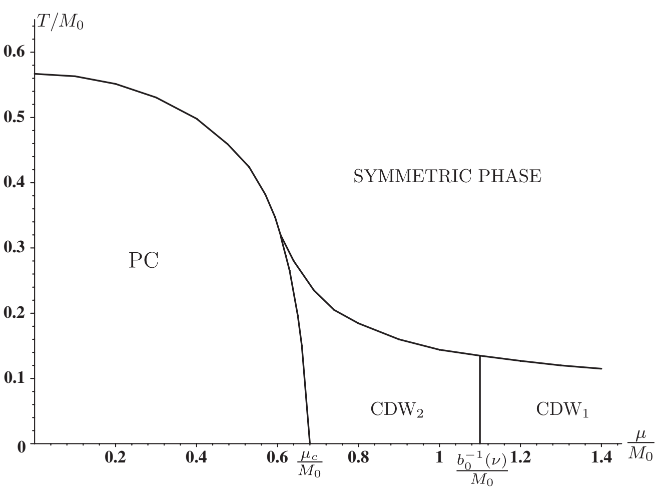

FIG. 9. The (μ, T) phase portrait at fixed ν, where $\nu_\alpha < \nu$. Here, $b_0^{-1}(\nu)$ is the function inverse to $b_0(\mu)$ defined in Fig. 3. PC denotes the homogeneous charged pion condensation phase with $M = 0$, $b = 0$, $\Delta = M_0$. CDW$_2$ denotes the inhomogeneous chiral density wave phase with $b < 0$. All critical curves do not depend on ν. Other notations are the same as in the previous figures.

to critical curves of the second order. It is interesting to remark that, for $0 < \nu < \nu_\alpha$ ($\nu_\alpha < \nu$), all critical curves of Fig. 8 (Fig. 9) do not depend on ν.

Finally, let us take $\mu_I = 0$ and compare the thermodynamical properties of our (1 + 1)-dimensional NJL model (1) (see the phase portrait of Fig. 8 at $\nu = 0$) with the corresponding massless (3 + 1)-dimensional NJL model with chiral $SU_L(2) \times SU_R(2)$ symmetry [29]. It turns out that, in the four-dimensional spacetime, in contrast to the (1 + 1)-dimensional case, a *second order phase transition* from a homogeneous chirally broken phase to an inhomogeneous one takes place. Moreover, depending on the value of the dynamical quark mass in vacuum, the inhomogeneous phase in the (3 + 1)-dimensional NJL model may occupy both a finite (compact) and infinite (noncompact) region of the (μ, T)-phase diagram, whereas in our two-dimensional NJL model (1) an inhomogeneous phase appears as a noncompact region (see Fig. 8).

VI. CONCLUSIONS

It is well-known that at nonzero baryon densities there might exist phases with a spatially inhomogeneous chiral condensate. This fact was established in the framework of both two-dimensional GN- or NJL-type models [17,18,26] and four-dimensional NJL-type models [27–31], where phases with a crystalline chiral condensate or with a CDW spatial inhomogeneity were proved to exist at nonzero values of the baryon chemical potential. Since the isotopic asymmetry of dense quark matter is an inevitable reality, a more adequate investigation of dense quark matter demands to include consideration of both baryon, μ, and isospin, μ_I, chemical potentials. In this paper and in contrast to previous papers [17,18,26], we study the possibility of spatially inhomogeneous chiral condensates in the $SU_L(2) \times SU_R(2)$ symmetric NJL$_2$ model (1) including the two above-mentioned chemical potentials in the leading order of the large-N_c expansion. For simplicity, the spatial inhomogeneity in our consideration is realized in the form of so-called chiral density waves or chiral spirals.

First, we have proven that at $\mu_I = 0$ and $T = 0$ the inhomogeneous CDW phase is realized in this $SU_L(2) \times SU_R(2)$ symmetric NJL$_2$ model only at sufficiently large values of μ, i.e. at $\mu > \mu_c \approx 0.68M_0$. (Here, M_0 is the dynamical quark mass in the vacuum, i.e. at zero values of chemical potentials.) In contrast, it is well-known that in the NJL$_2$ model with continuous $U_L(1) \times U_R(1)$ chiral symmetry the CDW phase appears at arbitrary nonzero values of $\mu > 0$ [17,18]. Moreover, it turns out that at $\mu_I = 0$ the Fermi surfaces of up/down quarks in the CDW phase are split by $2b_0(\mu)$, where $b_0(\mu)$ is the wave vector in this phase.

Second, if $\mu_I \neq 0$ and $T = 0$, then in the (μ_I, μ) phase diagram (see Fig. 7) the spatially inhomogeneous phases are allowed at $\mu > \mu_\alpha = \mu_c$ and arbitrary values of μ_I. This region is divided by the curve L into two domains. In one of them, each CDW is a clockwise twisted chiral spiral; in the other, it is a counterclockwise twisted spiral. The amplitude of chiral density waves does not depend on μ_I. The dependence of its wave vector b on μ and μ_I is defined by the formula (36). Since the isospin density n_I in these phases is equal to ν/π, we see that the wave vector b is linearly connected with n_I. In contrast, in the $U_L(1) \times U_R(1)$-symmetric NJL_2 model, the wave vector of the CDW phase is proportional to a quark number density [17,18]. The points of the curve L correspond to the spatially homogeneous phase (since here $b = 0$) with spontaneous chiral symmetry breaking. Indeed, the phase L is the residue of the homogeneous massive chirally nonsymmetric phase 2 of Fig. 1 which shrinks to L after taking into account inhomogeneity phenomena. Below the line $\mu = \mu_\alpha$, the homogeneous charged pion condensation phase is realized.

It turns out that at arbitrary μ_I-values in all above-mentioned inhomogeneous CDW phases as well as in the L phase, u-quarks are gapless excitations, but d-quarks are gapped ones.

Third, we have studied the influence of temperature on the formation of the CDW phases. In particular, it was shown that at $\mu_I = 0$ the (μ, T)-phase diagrams of the $SU_L(2) \times SU_R(2)$ and $U_L(1) \times U_R(1)$ symmetric NJL_2 models are quite different. Indeed, as was proved in [18], in the second model the CDW phase occupies in this diagram an infinite strip which includes points with arbitrary small μ values, whereas in the first model (see Fig. 8) the upper boundary of this phase is a monotonically decreasing function of μ. In addition, for rather small values of μ the CDW phase is forbidden in the framework of the $SU_L(2) \times SU_R(2)$ symmetric NJL_2 model.

We finally note that in this paper we have suggested a homogeneous pion condensate. It would be interesting in future to study the possibility of the spatially inhomogeneous pion condensation phase.

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[1] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).

[2] M. Buballa, Phys. Rep. **407**, 205 (2005); I. A. Shovkovy, Found. Phys. **35**, 1309 (2005); K. G. Klimenko and D. Ebert, Theor. Math. Phys. **150**, 82 (2007); M. G. Alford, A. Schmitt, K. Rajagopal, and T. Schäfer, Rev. Mod. Phys. **80**, 1455 (2008).

[3] E. S. Fraga and A. J. Mizher, Phys. Rev. D **78**, 025016 (2008); A. Ayala, A. Bashir, A. Raya, and A. Sanchez, Phys. Rev. D **80**, 036005 (2009); J. Phys. G **37**, 015001 (2010); A. J. Mizher, M. N. Chernodub, and E. S. Fraga, Phys. Rev. D **82**, 105016 (2010); M. N. Chernodub, Phys. Rev. D **82**, 085011 (2010); E. V. Gorbar, V. A. Miransky, and I. A. Shovkovy, Phys. Rev. D **83**, 085003 (2011).

[4] D. Blaschke, D. Ebert, K. G. Klimenko, M. K. Volkov, and V. L. Yudichev, Phys. Rev. D **70**, 014006 (2004); D. Ebert, K. G. Klimenko, and V. L. Yudichev, Phys. Rev. D **75**, 025024 (2007); Eur. Phys. J. C**53**, 65 (2007).

[5] H. J. Warringa, D. Boer, and J. O. Andersen, Phys. Rev. D **72**, 014015 (2005); K. Fukushima and H. J. Warringa, Phys. Rev. Lett. **100**, 032007 (2008).

[6] E. J. Ferrer, V. de la Incera, and C. Manuel, Nucl. Phys. **B747**, 88 (2006); E. J. Ferrer and V. de la Incera, Phys. Rev. D **76**, 045011 (2007); S. Fayazbakhsh and N. Sadooghi, Phys. Rev. D **82**, 045010 (2010); **83**, 025026 (2011).

[7] D. Ebert, K. G. Klimenko, M. A. Vdovichenko, and A. S. Vshivtsev, Phys. Rev. D **61**, 025005 (1999); D. Ebert and K. G. Klimenko, Nucl. Phys. **A728**, 203 (2003); T. Inagaki, D. Kimura, and T. Murata, Prog. Theor. Phys. Suppl. **153**, 321 (2004); E. S. Fraga and A. J. Mizher, Phys. Rev. D **78**, 025016 (2008); A. J. Mizher and E. S. Fraga, Nucl. Phys. **A831**, 91 (2009); J. K. Boomsma and D. Boer, Phys. Rev. D **81**, 074005 (2010); B. Chatterjee, H. Mishra, and A. Mishra, Phys. Rev. D, arXiv:1101.0498 [to be published].

[8] A. S. Vshivtsev, M. A. Vdovichenko, and K. G. Klimenko, J. Exp. Theor. Phys. **87**, 229 (1998); D. Ebert, K. G. Klimenko, A. V. Tyukov, and V. C. Zhukovsky, Eur. Phys. J. C **58**, 57 (2008).

[9] D. T. Son and M. A. Stephanov, Phys. At. Nucl. **64**, 834 (2001); M. Frank, M. Buballa, and M. Oertel, Phys. Lett. B **562**, 221 (2003); A. Barducci, R. Casalbuoni, G. Pettini, and L. Ravagli, Phys. Rev. D **69**, 096004 (2004); L. He and P. Zhuang, Phys. Lett. B **615**, 93 (2005); L. He, M. Jin, and P. Zhuang, Phys. Rev. D **71**, 116001 (2005); E. E. Svanes and J. O. Andersen, Nucl. Phys. **A857**, 16 (2011).

[10] D. Ebert and K. G. Klimenko, J. Phys. G **32**, 599 (2006); Eur. Phys. J. C**46**, 771 (2006); J. O. Andersen and L. Kyllingstad, J. Phys. G **37**, 015003 (2010); H. Abuki, R. Anglani, R. Gatto, M. Pellicoro, and M. Ruggieri, Phys. Rev. D **79**, 034032 (2009); H. Abuki, T. Brauner, and H. J. Warringa, Eur. Phys. J. C **64**, 123 (2009); C. f. Mu, L. y. He, and Y. x. Liu, Phys. Rev. D **82**, 056006 (2010).

[11] H. Abuki, M. Ciminale, R. Gatto, N. D. Ippolito, G. Nardulli, and M. Ruggieri, Phys. Rev. D **78**, 014002 (2008); H. Abuki, R. Anglani, R. Gatto, G. Nardulli,

and M. Ruggieri, Phys. Rev. D **78**, 034034 (2008); J. O. Andersen and T. Brauner, Phys. Rev. D **78**, 014030 (2008); M. Matsuzaki, Phys. Rev. D **82**, 016005 (2010).

[12] D. J. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).

[13] J. Feinberg, Ann. Phys. (N.Y.) **309**, 166 (2004); M. Thies, J. Phys. A **39**, 12707 (2006).

[14] U. Wolff, Phys. Lett. **157B**, 303 (1985); K. G. Klimenko, Theor. Math. Phys. **75**, 487 (1988); T. Inagaki, T. Kouno, and T. Muta, Int. J. Mod. Phys. A **10**, 2241 (1995); S. Kanemura and H.-T. Sato, Mod. Phys. Lett. A **10**, 1777 (1995); D. Fernandez-Fraile, Phys. Rev. D **83**, 065001 (2011).

[15] A. Barducci, R. Casalbuoni, M. Modugno, and G. Pettini, Phys. Rev. D **51**, 3042 (1995).

[16] A. Chodos, H. Minakata, F. Cooper, A. Singh, and W. Mao, Phys. Rev. D **61**, 045011 (2000); L. M. Abreu, A. P. C. Malbouisson, and J. M. C. Malbouisson, Europhys. Lett. **90**, 11001 (2010); Phys. Rev. D **83**, 025001 (2011).

[17] K. Ohwa, Phys. Rev. D **65**, 085040 (2002).

[18] V. Schon and M. Thies, arXiv:hep-th/0008175; Phys. Rev. D **62**, 096002 (2000); A. Brzoska and M. Thies, Phys. Rev. D **65**, 125001 (2002).

[19] S. K. Gandhi, H. F. Jones, and M. B. Pinto, Nucl. Phys. **B359**, 429 (1991); K. G. Klimenko, Z. Phys. C **60**, 677 (1993); J. L. Kneur, M. B. Pinto, and R. O. Ramos, Phys. Rev. D **74**, 125020 (2006); A. A. Osipov, B. Hiller, and A. H. Blin, Phys. Lett. B **653**, 346 (2007).

[20] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966); S. Coleman, Commun. Math. Phys. **31**, 259 (1973).

[21] A. Chodos and H. Minakata, Phys. Lett. A **191**, 39 (1994); H. Caldas, J. L. Kneur, M. B. Pinto, and R. O. Ramos, Phys. Rev. B **77**, 205109 (2008); H. Caldas, Nucl. Phys. **B807**, 651 (2009); J. Stat. Mech. (2010) P03027.

[22] D. Ebert, K. G. Klimenko, A. V. Tyukov, and V. C. Zhukovsky, Phys. Rev. D **78**, 045008 (2008); V. C. Zhukovsky, K. G. Klimenko, and T. G. Khunjua, Moscow Univ. Phys. Bull. **65**, 21 (2010).

[23] D. Ebert and K. G. Klimenko, Phys. Rev. D **80**, 125013 (2009).

[24] D. Ebert and K. G. Klimenko, arXiv:0902.1861.

[25] O. Schnetz, M. Thies, and K. Urlichs, Ann. Phys. (N.Y.) **314**, 425 (2004); G. Basar, G. V. Dunne, and M. Thies, Phys. Rev. D **79**, 105012 (2009); C. Boehmer and M. Thies, Phys. Rev. D **80**, 125038 (2009); J. Hofmann, Phys. Rev. D **82**, 125027 (2010).

[26] F. Correa, G. V. Dunne, and M. S. Plyushchay, Ann. Phys. (N.Y.) **324**, 2522 (2009); M. S. Plyushchay and L. M. Nieto, Phys. Rev. D **82**, 065022 (2010); G. Basar and G. V. Dunne, J. High Energy Phys. 01 (2011) 127; M. S. Plyushchay, A. Arancibia, and L. M. Nieto, Phys. Rev. D **83**, 065025 (2011).

[27] D. V. Deryagin, D. Y. Grigoriev, and V. A. Rubakov, Int. J. Mod. Phys. A **7**, 659 (1992); M. Sadzikowski and W. Broniowski, Phys. Lett. B **488**, 63 (2000).

[28] E. Nakano and T. Tatsumi, Phys. Rev. D **71**, 114006 (2005).

[29] D. Nickel, Phys. Rev. D **80**, 074025 (2009); S. Carignano, D. Nickel, and M. Buballa, Phys. Rev. D **82**, 054009 (2010).

[30] S. Maedan, Prog. Theor. Phys. **123**, 285 (2010).

[31] I. E. Frolov, K. G. Klimenko, and V. Ch. Zhukovsky, Phys. Rev. D **82**, 076002 (2010); Moscow Univ. Phys. Bull. **65**, 539 (2010).

[32] T. Kojo, Y. Hidaka, L. McLerran, and R. D. Pisarski, Nucl. Phys. **A843**, 37 (2010).T. Kojo, arXiv:1106.2187.

[33] G. Basar, G. V. Dunne, and D. E. Kharzeev, Phys. Rev. Lett. **104**, 232301 (2010).

[34] K. Fujikawa, Phys. Rev. D **21**, 2848 (1980).

[35] L. Jacobs, Phys. Rev. D **10**, 3956 (1974); W. Dittrich and B.-G. Englert, Nucl. Phys. **B179**, 85 (1981); K. G. Klimenko, Theor. Math. Phys. **70**, 87 (1987).