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Properties of the massive Gross-Neveu model with nonzero baryon and isospin chemical potentials

D. Ebert[1] and K. G. Klimenko[2]

[1]*Institute of Physics, Humboldt-University Berlin, 12489 Berlin, Germany*

[2]*IHEP and University of Dubna (Protvino branch), 142281 Protvino, Moscow Region, Russia*

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The properties of the two-flavored Gross-Neveu model with nonzero current quark mass are investigated in the (1 + 1)-dimensional space-time at finite isospin μ_I as well as quark number μ chemical potentials and zero temperature. The consideration is performed in the limit $N_c \to \infty$, i.e., in the case with an infinite number of colored quarks. In the plane of parameters μ_I, μ a rather rich phase structure is found, which contains phases with and without pion condensation. We have found a great variety of one-quark excitations of these phases, including gapless and gapped quasiparticles. Moreover, the mesonic mass spectrum in each phase is also investigated.

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I. INTRODUCTION

During the last decade great attention was paid to the investigation of the QCD phase diagram in terms of baryonic as well as isotopic (isospin) chemical potentials. First of all, this interest is motivated by experiments on heavy-ion collisions, where we have to deal with dense baryonic matter which has an evident isospin asymmetry, i.e., different neutron and proton contents of initial ions. Moreover, the dense hadronic/quark matter inside compact stars is also isotopically asymmetric. Generally speaking, one of the important QCD applications is just to describe dense and hot baryonic matter. However, in the above-mentioned realistic situations the density is rather small, and weak coupling QCD analysis is not applicable. So, different nonperturbative methods or effective theories such as chiral effective Lagrangians and especially Nambu-Jona-Lasinio (NJL) type models [1] are usually employed for the consideration of the properties of dense and hot baryonic matter under heavy-ion experimental and/or compact star conditions, i.e., in the presence of such external conditions as temperature and chemical potentials, magnetic field, finite size effects etc. (see, e.g., the papers [2–9] and references therein). In particular, the color superconductivity [4,5] as well as parity violation and charged pion condensation [10–15] phenomena of dense quark matter were investigated in the framework of these QCD-like effective models.

It is necessary to note that an effective description of QCD in terms of NJL models, i.e., through an employment of four-fermion theories in (3 + 1)-dimensional space-time, is usually valid only at *rather low* energies and densities. Besides, at present time there is the consensus that another class of theories, the set of (1 + 1)-dimensional Gross-Neveu (GN) type models [16,17], can also be used for a reasonable qualitative consideration of the QCD properties *without any restrictions* on the energy/density values, which is in an encouraging contrast with NJL models. Indeed, the GN-type models are renormalizable; the asymptotic freedom and spontaneous chiral symmetry breaking are other properties inherent both for QCD and GN theories, etc. In addition, the $\mu - T$ phase diagram is qualitatively the same in the QCD and GN model [18–22] (here μ is the quark number chemical potential and T is the temperature). Note also that GN-type models are suitable for the description of physics in quasi one-dimensional condensed matter systems like polyacetylene [23]. Thus, due to the relative simplicity of GN models in the leading order of the large N_c-expansion (N_c is the number of colored quarks), their usage is convenient for the application of nonperturbative methods in quantum field theory [24].

Before investigating different physical effects relevant to a real (3 + 1)-dimensional world in the framework of two-dimensional GN models, let us recall that there is a no-go theorem by Mermin-Wagner-Coleman forbidding the spontaneous breaking of continuous symmetries in two dimensions [25]. This theorem is based on the fact that in (1 + 1)-dimensional space-time the Green function (correlator) of two scalar fields has at large distances a behavior $|x - y|^{-1/N_c}$. Thus, if we take the limit $|x - y| \to \infty$ first, the correlator vanishes at finite N_c and, according to the cluster property, we formally obtain a zero vacuum expectation value of the scalar field, i.e. a prohibition of spontaneous symmetry breaking. However, there is a way to overcome this no-go theorem. Indeed, if the limit $N_c \to \infty$ is taken first, then for $|x - y| \to \infty$ we formally obtain a nonzero vacuum expectation value for the scalar field, i.e., the possibility for spontaneous symmetry breaking. It means that just the leading order of the large N_c approximation supplies us in any (1 + 1)-dimensional model with a consistent field theory in which spontaneous symmetry breaking might occur. At present time this fact is well understood (see, e.g., the discussion in [20–22]). This result restricts the range of validity of the no-go theorem to the finite N_c-case only. Clearly, since the no-go theorem

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does not work in the limit $N_c \to \infty$, the investigation of any low-dimensional model in the leading order at $N_c \to \infty$ is much more physically appealing than the consideration of the model at finite N_c.

By this reason, such phenomena of dense QCD as color superconductivity (spontaneous breaking of the color symmetry) or charged pion condensation (spontaneous breaking of the continuous isospin symmetry) might be simulated in terms of simpler (1 + 1)-dimensional GN-type models in the leading order of the large N_c approximation (see, e.g., [21,26], correspondingly).

In our previous paper [26] the phase diagram of the (1 + 1)-dimensional GN model with two massless quark flavors was investigated under the constraint that quark matter occupies a finite space volume (see also the relevant papers [27]). In particular, the charged pion condensation phenomenon in cold quark matter with zero baryonic density, i.e., at $\mu = 0$, but nonzero isotopic density, i.e., with nonzero isospin chemical potential μ_I, was studied there in the large N_c-limit. In contrast, in the present paper we consider, in the leading order of the $1/N_c$-expansion, the phase portrait of the above-mentioned massive GN model in a more general case, where, for simplicity, the temperature is taken to be zero, but both isospin and quark number chemical potentials are nonzero, i.e., $\mu_I \neq 0$ and $\mu \neq 0$, and the space-time is considered to have the usual topology, $R^1 \times R^1$. Our consideration is based on the case of homogeneous condensates (an extension to inhomogeneous condensates in the case of $\mu_I = 0$ was recently considered in [22,28]). We suppose that these investigations will shed some new light on the physics of cold dense and isotopically asymmetric quark matter which might exist in compact stars, where baryon density is obviously nonzero (i.e., $\mu \neq 0$).

The paper is organized as follows. In Secs. II and III the effective action and thermodynamic potential of the two-flavored massive Gross-Neveu model are obtained in the presence of a quark number as well as isotopic chemical potentials. In Sec. IV the phase structure of the model is investigated both in different particular cases ($\mu \neq 0$, $\mu_I = 0$ etc.) and in the general case of $\mu \neq 0$, $\mu_I \neq 0$. It turns out that at $\mu_I \neq 0$ and rather small values of μ, the gapped pion condensed phase (PC) occurs. However, at larger values of μ several normal dense quark matter phases (without PC) are found to exist with different quasiparticle excitation properties of their ground states. In Sec. V the meson mass spectrum of each phase is discussed. Some technical details concerning the effective action and quark propagator are relegated to two Appendices.

II. THE MODEL AND ITS EFFECTIVE ACTION

We consider a (1 + 1)-dimensional model which describes dense quark matter with two massive quark flavors (u and d quarks). Its Lagrangian has the form

$$L = \bar{q}\left[\gamma^\nu i\partial_\nu - m_0 + \mu\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0\right]q + \frac{G}{N_c}[(\bar{q}q)^2 + (\bar{q}i\gamma^5\vec{\tau}q)^2], \quad (1)$$

where the quark field $q(x) \equiv q_{i\alpha}(x)$ is a flavor doublet ($i = 1, 2$ or $i = u, d$) and color N_c-plet ($\alpha = 1, \ldots, N_c$) as well as a two-component Dirac spinor (the summation in (1) over flavor, color, and spinor indices is implied); τ_k ($k = 1, 2, 3$) are Pauli matrices; the quark number chemical potential μ in (1) is responsible for the nonzero baryonic density of quark matter, whereas the isospin chemical potential μ_I is taken into account in order to study properties of quark matter at nonzero isospin densities (in this case the densities of u and d quarks are different). Evidently, the model (1) is a simple generalization of the original (1 + 1)-dimensional Gross-Neveu model [16] with a single massless quark color N_c-plet to the case of two massive quark flavors and additional chemical potentials. As a result, in the case under consideration we have a modified flavor symmetry group, which depends essentially on whether the bare quark mass m_0 and isospin chemical potential μ_I take zero or nonzero values. Indeed, at $\mu_I = 0$, $m_0 = 0$ the Lagrangian (1) is invariant under transformations from the chiral $SU_L(2) \times SU_R(2)$ group. Then, at $\mu_I \neq 0$, $m_0 = 0$ this symmetry is reduced to $U_{I_3L}(1) \times U_{I_3R}(1)$, where $I_3 = \tau_3/2$ is the third component of the isospin operator (here and above the subscripts L, R mean that the corresponding group acts only on left-, right-handed spinors, respectively). Evidently, this symmetry can also be presented as $U_{I_3}(1) \times U_{AI_3}(1)$, where $U_{I_3}(1)$ is the isospin subgroup and $U_{AI_3}(1)$ is the axial isospin subgroup. Quarks are transformed under these subgroups as $q \to \exp(i\alpha\tau_3)q$ and $q \to \exp(i\alpha\gamma^5\tau_3)q$, respectively. In the case $m_0 \neq 0$, $\mu_I = 0$ the Lagrangian (1) is invariant with respect to the $SU_I(2)$, which is a diagonal subgroup of the chiral $SU_L(2) \times SU_R(2)$ group. Finally, in the most general case with $m_0 \neq 0$, $\mu_I \neq 0$ the initial model (1) is symmetric under the above-mentioned isospin subgroup $U_{I_3}(1)$. In addition, in all foregoing cases the model is color $SU(N_c)$ invariant.

The linearized version of the Lagrangian (1), which contains composite bosonic fields $\sigma(x)$ and $\pi_a(x)$ ($a = 1, 2, 3$), has the following form:

$$\tilde{L} = \bar{q}\left[\gamma^\nu i\partial_\nu - m_0 + \mu\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0 - \sigma - i\gamma^5\pi_a\tau_a\right]q - \frac{N_c}{4G}[\sigma\sigma + \pi_a\pi_a]. \quad (2)$$

From the Lagrangian (2) one gets the following constraint equations for the bosonic fields

$$\sigma(x) = -2\frac{G}{N_c}(\bar{q}q); \qquad \pi_a(x) = -2\frac{G}{N_c}(\bar{q}i\gamma^5\tau_a q). \quad (3)$$

Obviously, the Lagrangian (2) is equivalent to the

Lagrangian (1) when using the constraint Eqs. (3). Furthermore, it is clear that the bosonic fields (3) are transforming under the isospin $U_{I_3}(1)$ subgroup in the following manner:

$$\begin{aligned} U_{I_3}(1)\colon\ \sigma &\to \sigma; \qquad \pi_3 \to \pi_3; \\ \pi_1 &\to \cos(2\alpha)\pi_1 + \sin(2\alpha)\pi_2; \\ \pi_2 &\to \cos(2\alpha)\pi_2 - \sin(2\alpha)\pi_1, \end{aligned} \tag{4}$$

i.e., the expression $(\pi_1^2 + \pi_2^2)$ remains unchanged under an action of the isospin subgroup $U_{I_3}(1)$.

There is a common footing for obtaining both the thermodynamic potential and one-particle irreducible Green functions of bosonic $\sigma(x)$ and $\pi_a(x)$ fields (3) which is based on the effective action $\mathcal{S}_{\rm eff}(\sigma, \pi_a)$ of the model. In the leading order of the large N_c-expansion (corresponding to the one fermion-loop or mean field approximation), this quantity is defined in terms of the Lagrangian (2) through the relation

$$\exp(i\mathcal{S}_{\rm eff}(\sigma, \pi_a)) = N' \int [d\bar{q}][dq] \exp(i \int \tilde{L} d^2x), \tag{5}$$

where N' is a normalization constant. It is clear from (2) and (5) that

$$\mathcal{S}_{\rm eff}(\sigma, \pi_a) = -N_c \int \frac{\sigma^2 + \pi_a^2}{4G} d^2x + \tilde{\mathcal{S}}_{\rm eff}, \tag{6}$$

where the quark contribution to the effective action, i.e., the term $\tilde{\mathcal{S}}_{\rm eff}$ in (6), is given by

$$\exp(i\tilde{\mathcal{S}}_{\rm eff}) = N' \int [d\bar{q}][dq] \exp\left(i \int \bar{q} D q d^2x\right) = \det D. \tag{7}$$

Here we used the notations

$$D = i\gamma^\nu \partial_\nu - m_0 + \mu\gamma^0 + \nu\tau_3\gamma^0 - \sigma - i\gamma^5\pi_a\tau_a \tag{8}$$

and $\nu = \mu_I/2$. Note also that D is a nontrivial operator in coordinate (x), spinor (s), and flavor (f) spaces, but it is proportional to the unit operator in the N_c-dimensional color (c) space. Then, using the general formula $\det D = \exp \mathrm{Tr}_{xcfs} \ln D$, one obtains the following expression for the effective action:

$$\mathcal{S}_{\rm eff}(\sigma, \pi_a) = -N_c \int \frac{\sigma^2 + \pi_a^2}{4G} d^2x - iN_c \,\mathrm{Tr}_{sfx} \ln D, \tag{9}$$

where we have taken into account that the trace of the operator $\ln D$ over the color space is proportional to N_c.

Starting from (9), one can define the thermodynamic potential (TDP) of the model in the mean-field approximation:

$$\mathcal{S}_{\rm eff}|_{(\sigma,\pi_a = \rm const)} = -N_c \Omega_{\mu,\nu}(\sigma, \pi_a) \int d^2x. \tag{10}$$

The ground state expectation values (mean values) of the bosonic fields, $\langle\sigma(x)\rangle \equiv \sigma^o$ and $\langle\pi_a(x)\rangle \equiv \pi_a^o$, are solutions of the gap equations for the TDP $\Omega_{\mu,\nu}(\sigma, \pi_a)$ (in our approach all ground state expectation values do not depend on coordinates x):

$$\frac{\partial\Omega_{\mu,\nu}}{\partial\sigma} = 0, \qquad \frac{\partial\Omega_{\mu,\nu}}{\partial\pi_a} = 0, \quad \text{where } a = 1, 2, 3. \tag{11}$$

In particular, it follows from (11) that if $m_0 \neq 0$ then $\pi_3^o = 0$. In addition, one can put $\pi_2^o = 0$, since the effective action depends on π_1 and π_2 fields through the combination $(\pi_1^2 + \pi_2^2)$. Next, let us perform the following shift of bosonic fields in (9): $\sigma(x) \to \sigma(x) + \sigma^o$, $\pi_1(x) \to \pi_1(x) + \pi_1^o$, whereas the other bosonic fields, $\pi_{2,3}$, stay unshifted. (Obviously, after shifting the new bosonic fields $\sigma(x)$, $\pi_a(x)$ now denote the small quantum fluctuations around the mean values σ^o, π_a^o of mesons rather than the original fields (3)). Moreover, we use the notations $\sigma^o \equiv M - m_0$ and $\pi_1^o \equiv \Delta$. In this case

$$\begin{aligned} D &= (i\gamma^\nu\partial_\nu - M + \mu\gamma^0 + \nu\tau_3\gamma^0 - i\gamma^5\Delta\tau_1) \\ &\quad - (\sigma(x) + i\gamma^5\pi_a(x)\tau_a) \\ &\equiv S_0^{-1} - (\sigma(x) + i\gamma^5\pi_a(x)\tau_a), \end{aligned} \tag{12}$$

where S_0 is the quark propagator which is a 2×2 matrix in the flavor space, presented in Appendix B. Then, expanding the obtained expression into a Taylor-series up to second order of small bosonic fluctuations $\sigma(x)$, $\pi_a(x)$, we have

$$\mathcal{S}_{\rm eff}(\sigma, \pi_a) = \mathcal{S}_{\rm eff}^{(0)} + \mathcal{S}_{\rm eff}^{(2)}(\sigma, \pi_a) + \cdots, \tag{13}$$

where (due to the gap equations, the linear term in meson and diquark fields is absent in (13))

$$\begin{aligned} \frac{1}{N_c}\mathcal{S}_{\rm eff}^{(0)} &= -\int d^2x \frac{(M - m_0)^2 + \Delta^2}{4G} - i\,\mathrm{Tr}_{sfx} \ln(S_0^{-1}) \\ &\equiv -\Omega_{\mu,\nu}(M, \Delta) \int d^2x, \end{aligned} \tag{14}$$

$$\begin{aligned} \frac{1}{N_c}\mathcal{S}_{\rm eff}^{(2)}(\sigma, \pi_a) &= -\int d^2x \frac{\sigma^2 + \pi_a^2}{4G} \\ &\quad + \frac{i}{2}\,\mathrm{Tr}_{sfx}\{S_0(\sigma + i\gamma^5\pi_a\tau_a) \\ &\quad \times S_0(\sigma + i\gamma^5\pi_a\tau_a)\}. \end{aligned} \tag{15}$$

The TDP $\Omega_{\mu,\nu}(M, \Delta)$ from (14) will be calculated in the next section, where on the basis of this function the phase structure of the GN model (1) in the leading order over $1/N_c$ is considered. Note also that in (13) and (15) the bosonic fluctuation fields σ, π_a are really the coordinate dependent quantities. The trace of the S_0-operator and the products of σ, π_a-fields in (15) should be understood in the sense of formula (A2) (see Appendix A). Note the remarkable property that the effective action (15) is a generating

functional of two-point and one-particle irreducible (1PI) Green functions of σ- and π-mesons. Indeed:

$$\Gamma_{XY}(x-y) = -\frac{\delta^2 \mathcal{S}^{(2)}_{\rm eff}}{\delta Y(y)\delta X(x)}, \tag{16}$$

where $X(x)$, $Y(x) = \sigma(x)$, $\pi_a(x)$, and $\Gamma_{XY}(x-y)$ is the 1PI Green function of the fields $X(x)$, $Y(x)$. Variational derivatives in (16) should be taken in accordance with the general formula (A3) (see Appendix A). In the following, on the basis of these Green functions we study the meson mass spectrum in different phases of the model.

III. THERMODYNAMIC POTENTIAL

The Fourier transformation $\bar{S}_0^{-1}(p)$ of the inverse quark propagator S_0^{-1} (12) has the form:

$$\bar{S}_0^{-1}(p) = \not{p} + \mu\gamma^0 + \nu\tau_3\gamma^0 - M - i\gamma^5\Delta\tau_1. \tag{17}$$

Clearly, in the direct product of spinor and flavor spaces it is a 4×4 matrix, which has four eigenvalues:

$$\epsilon_{1,2,3,4} = -M \pm\sqrt{(p_0+\mu)^2 - p_1^2 - \Delta^2 + \nu^2 \pm 2\nu\sqrt{(p_0+\mu)^2-\Delta^2}}. \tag{18}$$

Then, applying the general formula (A5) to the expression (14) for the thermodynamic potential, one gets:

$$\begin{aligned}\Omega_{\mu,\nu}(M,\Delta) &= \frac{(M-m_0)^2+\Delta^2}{4G} + i\sum_{i=1}^{4}\int\frac{d^2p}{(2\pi)^2}\ln(\epsilon_i)\\ &= \frac{(M-m_0)^2+\Delta^2}{4G} + i\int\frac{d^2p}{(2\pi)^2}\\ &\quad\times\ln\{[(p_0+\mu)^2-(E^+_\Delta)^2]\\ &\quad\times[(p_0+\mu)^2-(E^-_\Delta)^2]\},\end{aligned} \tag{19}$$

where $E^\pm_\Delta = \sqrt{(E^\pm)^2+\Delta^2}$, $E^\pm = E\pm\nu$, $\nu = \mu_I/2$ and $E = \sqrt{p_1^2+M^2}$. The system of gap equations directly follows from (19):

$$\begin{aligned}0 &= \frac{\partial\Omega_{\mu,\nu}(M,\Delta)}{\partial M}\\ &\equiv \frac{M-m_0}{2G} - 2iM\int\frac{d^2p}{(2\pi)^2E}\left\{\frac{E^+}{(p_0+\mu)^2-(E^+_\Delta)^2}\right.\\ &\quad\left.+\frac{E^-}{(p_0+\mu)^2-(E^-_\Delta)^2}\right\},\\ 0 &= \frac{\partial\Omega_{\mu,\nu}(M,\Delta)}{\partial\Delta}\\ &\equiv \frac{\Delta}{2G} - 2i\Delta\int\frac{d^2p}{(2\pi)^2}\left\{\frac{1}{(p_0+\mu)^2-(E^+_\Delta)^2}\right.\\ &\quad\left.+\frac{1}{(p_0+\mu)^2-(E^-_\Delta)^2}\right\}.\end{aligned} \tag{20}$$

The TDP $\Omega_{\mu,\nu}(M,\Delta)$ is symmetric under the transformations $\mu\to-\mu$ and/or $\mu_I\to-\mu_I$. Hence, it is sufficient to consider only the region $\mu\geq 0$, $\mu_I\geq 0$. In this case, one can integrate in (19) over p_0 with the help of the formula

$$\int\frac{dp_0}{2\pi}\ln[(p_0+a)^2-b^2] = \frac{i}{2}\{|a-b|+|a+b|\}$$

(which is valid up to an infinite constant independent of quantities a, b) and obtain:

$$\begin{aligned}\Omega_{\mu,\nu}(M,\Delta) &= \frac{(M-m_0)^2+\Delta^2}{4G} - \int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\{E^+_\Delta + E^-_\Delta\\ &\quad+(\mu-E^+_\Delta)\theta(\mu-E^+_\Delta)\\ &\quad+(\mu-E^-_\Delta)\theta(\mu-E^-_\Delta)\},\end{aligned} \tag{21}$$

where $\theta(x)$ is the Heaviside theta-function. In a similar way, the system of gap Eqs. (20) is transformed to the following one:

$$\begin{aligned}0 &= \frac{\partial\Omega_{\mu,\nu}(M,\Delta)}{\partial M}\\ &\equiv \frac{M-m_0}{2G} - M\int_{-\infty}^{\infty}\frac{dp_1}{2\pi E}\left\{\frac{\theta(E^+_\Delta-\mu)E^+}{E^+_\Delta}\right.\\ &\quad\left.+\frac{\theta(E^-_\Delta-\mu)E^-}{E^-_\Delta}\right\},\end{aligned} \tag{22}$$

$$\begin{aligned}0 &= \frac{\partial\Omega_{\mu,\nu}(M,\Delta)}{\partial\Delta}\\ &\equiv \frac{\Delta}{2G} - \Delta\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\left\{\frac{\theta(E^+_\Delta-\mu)}{E^+_\Delta} + \frac{\theta(E^-_\Delta-\mu)}{E^-_\Delta}\right\}.\end{aligned} \tag{23}$$

The coordinates (gap values) M and Δ of the global minimum point of the TDP (21) supply us with two ground state expectation values $\langle\bar{q}q\rangle$ and $\langle\bar{q}i\gamma^5\tau_1 q\rangle$, respectively, through the relations $M = m_0 + \langle\sigma\rangle$, $\Delta = \langle\pi_1\rangle$ and formulas (3). In particular, if the gap Δ is equal to zero, the ground state of the model is isotopically symmetric and there is no condensation of charged pions. However, if $\Delta\neq 0$, then the ground state describes the phase with charged pion condensation, where the isospin $U_{I_3}(1)$ symmetry is spontaneously broken. In this phase the space parity is also spontaneously broken. Note also that the physical essence of the other gap M is the dynamical quark mass which is not equal to the bare mass m_0, evidently.

It is clear that the TDP (21) is an ultraviolet divergent quantity, so one should renormalize it, using a special dependence of the bare quantities such as the bare coupling constant G and the bare quark mass m_0 on the cutoff parameter Λ (Λ restricts the integration region in the divergent integrals, $|p_1|<\Lambda$). The renormalization procedure for the simplest massive GN model was already discussed in the literature, see, e.g., in [19,20,29]. In a similar way, it is easy to see that, cutting of the divergent integral in (21) and using the substitution $G\equiv G(\Lambda)$ and $m_0\equiv mG(\Lambda)$, where

$$\frac{1}{2G(\Lambda)} = \frac{1}{\pi}\int_{-\Lambda}^{\Lambda} dp_1 \frac{1}{\sqrt{M_0^2 + p_1^2}} = \frac{2}{\pi}\ln\left(\frac{\Lambda + \sqrt{M_0^2 + \Lambda^2}}{M_0}\right) \tag{24}$$

and m is a new free finite renormalization-invariant massive parameter[1] (which does not depend on the cutoff Λ), it is possible to obtain for the TDP (21) a finite renormalization-invariant expression. Namely,

$$\Omega_{\mu,\nu}(M,\Delta) = \lim_{\Lambda\to\infty}\Omega_{\mu,\nu}(M,\Delta;\Lambda), \tag{25}$$

where

$$\Omega_{\mu,\nu}(M,\Delta;\Lambda) = \frac{M^2+\Delta^2}{4G(\Lambda)} - \frac{mM}{2} - \int_{-\Lambda}^{\Lambda}\frac{dp_1}{2\pi} \times \{E_\Delta^+ + E_\Delta^- + (\mu - E_\Delta^+)\theta(\mu - E_\Delta^+) + (\mu - E_\Delta^-)\theta(\mu - E_\Delta^-)\} + \frac{\Lambda^2}{\pi}. \tag{26}$$

(To obtain (26) we have omitted the unessential constant $\frac{m_0^2}{4G}$ as well as added another one, $\frac{\Lambda^2}{\pi}$.) In (24) the cutoff independent quantity M_0 is the dynamically generated quark mass in the vacuum, i.e. at $\mu = 0$ and $\mu_I = 0$, taken in the chiral limit, i.e. at $m_0 = 0$ (see below). (The renormalized expressions for the gap equations are obtained in the limit $\Lambda \to \infty$, if the replacements $G \to G(\Lambda)$, $m_0 \to mG(\Lambda)$ and $|p_1| < \Lambda$ are done in (22) and (23), or by a direct differentiation of the expression (25).) The expression (26) can also be presented in the alternative form

$$\Omega_{\mu,\nu}(M,\Delta;\Lambda) = V_0(M,\Delta;\Lambda) - \frac{mM}{2} - \int_{-\Lambda}^{\Lambda}\frac{dp_1}{2\pi} \times \{E_\Delta^+ + E_\Delta^- - 2\sqrt{p_1^2 + M^2 + \Delta^2} + (\mu - E_\Delta^+)\theta(\mu - E_\Delta^+) + (\mu - E_\Delta^-)\theta(\mu - E_\Delta^-)\}, \tag{27}$$

where

$$V_0(M,\Delta;\Lambda) = \frac{M^2+\Delta^2}{4G(\Lambda)} - \frac{1}{\pi}\int_{-\Lambda}^{\Lambda} dp_1\sqrt{p_1^2 + M^2 + \Delta^2} + \frac{\Lambda^2}{\pi}. \tag{28}$$

Obviously, the integral in (27) is convergent at $\Lambda \to \infty$. Since

$$\lim_{\Lambda\to\infty} V_0(M,\Delta;\Lambda) = \frac{M^2+\Delta^2}{2\pi}\left[\ln\left(\frac{M^2+\Delta^2}{M_0^2}\right) - 1\right] \equiv V_0(M,\Delta), \tag{29}$$

one can easily obtain from (25), (27), and (29) the following finite renormalization-invariant expression for the TDP:

$$\Omega_{\mu,\nu}(M,\Delta) = V_0(M,\Delta) - \frac{mM}{2} - \int_{-\infty}^{\infty}\frac{dp_1}{2\pi} \times \{E_\Delta^+ + E_\Delta^- - 2\sqrt{p_1^2 + M^2 + \Delta^2} + (\mu - E_\Delta^+)\theta(\mu - E_\Delta^+) + (\mu - E_\Delta^-)\theta(\mu - E_\Delta^-)\}. \tag{30}$$

Note that the integral in (30) is convergent. In the particular case of $\mu = 0$, $\mu_I = 0$, and $m = 0$, i.e., for the massless GN model in the vacuum, it follows from (30):

$$\Omega_{\mu,\nu}(M,\Delta)|_{\mu=0,\nu=0,m=0} = \frac{M^2+\Delta^2}{2\pi}\left[\ln\left(\frac{M^2+\Delta^2}{M_0^2}\right) - 1\right]. \tag{31}$$

Since for a strongly interacting system the space-parity in the vacuum is expected to be a conserved quantity, we put Δ equal to zero in (31). As a result, the global minimum of the TDP (31) lies in the point $M = M_0$, which means that in the vacuum and at $m_0 = 0$ the dynamically generated quark mass is just the parameter M_0 introduced in (24). However, in the general case, i.e., at nonzero values of the chemical potentials, the dynamical quark mass depends certainly on μ, μ_I and obeys the system of the gap Eqs. (22) and (23) (or (20)). Another free parameter of the massive GN model, the quantity m, is not directly related to the quark mass, but rather to the mass of π-mesons.

In the following, when studying the phase structure or the meson mass spectrum, the quantity M_0 is still treated as a free parameter, however the massive parameter of the model, $m \equiv \alpha M_0/\pi$, is fixed by $\alpha = \alpha_0 \approx 0.17$. In this case the vacuum properties of the massive GN model resemble the situation in some NJL-type models in realistic (3 + 1)-space-time (for a more detailed discussion, see Sec. IV C).

For the forthcoming investigations we need also the expressions for the density of quark number n_q and isospin density n_I:

$$n_q \equiv -\frac{\partial\Omega_{\mu,\nu}}{\partial\mu} = \int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\{\theta(\mu - E_\Delta^+) + \theta(\mu - E_\Delta^-)\}, \tag{32}$$

[1]Note, the quantity m does not equal to the physical or dynamical quark mass M. The last one is defined by the pole position of the quark propagator. Alternatively, it can be found as a gap, i.e., one of the coordinates of the global minimum point of the thermodynamic potential (see also the remark in the paragraph just after (23)).

$$n_I \equiv -\frac{\partial \Omega_{\mu,\nu}}{2\partial \nu} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\left\{\frac{E+\nu}{E_\Delta^+}\theta(E_\Delta^+ - \mu) - \frac{E-\nu}{E_\Delta^-}\theta(E_\Delta^- - \mu)\right\}. \tag{33}$$

IV. PHASE STRUCTURE OF THE MODEL

A. Particular case: $\mu = 0$, $\mu_I = 0$

Introducing the notation $m \equiv \alpha M_0/\pi$, one can get from (30) the following expression for the TDP at $\mu = 0$, $\mu_I = 0$ (usually, this quantity is called effective potential):

$$\Omega_0(M,\Delta) = \frac{M^2+\Delta^2}{2\pi}\left[\ln\left(\frac{M^2+\Delta^2}{M_0^2}\right) - 1\right] - \frac{\alpha M_0 M}{2\pi}. \tag{34}$$

The corresponding gap equations look like

$$\frac{2\pi\partial\Omega_0(M,\Delta)}{\partial M} \equiv 2M\ln\left(\frac{M^2+\Delta^2}{M_0^2}\right) - \alpha M_0 = 0, \tag{35}$$

$$\frac{2\pi\partial\Omega_0(M,\Delta)}{\partial \Delta} \equiv 2\Delta\ln\left(\frac{M^2+\Delta^2}{M_0^2}\right) = 0. \tag{36}$$

The gap system (35) and (36) has several solutions, but the global minimum point (GMP) of the TDP (34) corresponds to the value $\Delta = 0$. Then, at $\Delta = 0$, the Eq. (35) vs M has three solutions of different signs. Just the one with largest absolute value corresponds to the GMP of the TDP. This quantity (gap) is denoted by M and depicted in Fig. 1 as a function of the variable α. Since the quark number density n_q and isospin density n_I (32) and (33) are equal to zero in this GMP, we conclude that at $\mu = 0$ and $\mu_I = 0$ the ground state of the model corresponds to the empty space, i.e. to the vacuum. Hence, in this case the gap M is the dynamical quark mass in the vacuum. Clearly, the gap M coincides with M_0 in the chiral limit, $\alpha = 0$. In addition, in Fig. 1 the behavior of the π-meson mass M_π vs α in the case of $\mu = 0$ and $\mu_I = 0$ is also presented (it is the solution of the Eq. (57) from Sec. V A 1). From the investigations of Sec. IV C it will become clear that M_π coincides with the critical value μ_{Ic} of the isotopical chemical potential μ_I, at which the system passes from the vacuum state to the pion condensed phase. Just this fact is reflected in Fig. 1. Moreover, we have also depicted in this figure the behavior of the critical value μ_c vs α of the chemical potential μ, at which the system passes from the vacuum to the normal quark matter phase at $\nu = 0$ (see Sec. IV B).

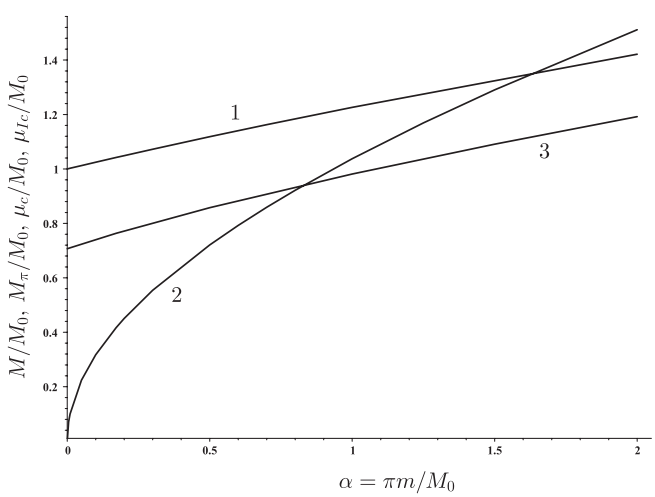

FIG. 1. Dynamical quark mass M (curve 1) and π-meson mass M_π (curve 2) vs $\alpha \equiv \pi m/M_0$ at $\mu = 0$, $\mu_I = 0$. Curve 3 is the critical value μ_c of the vacuum–normal quark matter phase transition (at $\mu_I = 0$); the critical value μ_{Ic} of the vacuum–PC phase transition is also given by the curve 2, i.e., $\mu_{Ic} = M_\pi$ (see Sec. IV C).

It is easily seen from Fig. 1 that the relation between the gap M in the vacuum and the pion mass M_π (at $\mu = 0$ and $\mu_I = 0$) has a strong α-dependency and for some values of this parameter does not describe real physics. Recall, in real (3 + 1)-dimensional physical models the dynamical quark mass M is usually greater than M_π at $\mu = 0$ and $\mu_I = 0$ and depends on the model parameters (coupling constants, cutoff parameter, etc.). In particular, the values $M = 350$ MeV and $M_\pi = 140$ MeV, i.e., $M/M_\pi = 5/2$, are often used in the NJL model investigations of dense quark matter [30]. So, in the following consideration of the phase structure of the model (1) and its meson mass spectrum in the most general case of $\mu \neq 0$ and $\nu \neq 0$, we will suppose the same relation between M and M_π at $\mu = 0$ and $\mu_I = 0$. Evidently (see Fig. 1), this choice corresponds to $\alpha = \alpha_0 \approx 0.17$. Having fixed the parameter $\alpha = \alpha_0 \approx 0.17$, it is then possible to obtain $M/M_0 \approx 1.04$, $M_\pi/M_0 \approx 0.42$, and $m/M_0 \approx 0.05$, where M_0 is the dynamical quark mass in the massless GN model at $\mu = 0$ and $\mu_I = 0$.

B. Particular case: $\mu \neq 0$, $\mu_I = 0$

Using again the notation $m \equiv \alpha M_0/\pi$, one can get from (30) the following expression for the TDP at $\mu \neq 0$, $\mu_I = 0$:

$$\Omega_\mu(M,\Delta) = \frac{M^2+\Delta^2}{2\pi}\left[\ln\left(\frac{M^2+\Delta^2}{M_0^2}\right) - 1\right] - \frac{\alpha M_0 M}{2\pi} + \frac{\theta(\mu - \sqrt{M^2+\Delta^2})}{\pi}\left[(M^2+\Delta^2) \times \ln\left(\frac{\mu + \sqrt{\mu^2 - M^2 - \Delta^2}}{\sqrt{M^2+\Delta^2}}\right) - \mu\sqrt{\mu^2 - M^2 - \Delta^2}\right]. \tag{37}$$

It follows from the gap equations for the TDP (37) that $\Delta = 0$ in its global minimum point, whereas the

M-coordinate of the GMP obeys the equation:

$$\theta(\mu^2 - M^2)\ln\frac{(\mu + \sqrt{\mu^2 - M^2})^2}{M_0^2} + \theta(M^2 - \mu^2)\ln\frac{M^2}{M_0^2} = \frac{\alpha M_0}{2M}. \quad (38)$$

Studying the GMP of the TDP (37) with the help of the stationary Eq. (38), it is possible to show that at $\mu < \mu_c$ the GMP is arranged in the point $(M, \Delta = 0)$, where both the critical value μ_c and the gap M are depicted in Fig. 1. In this case the system is arranged in the vacuum state with $n_q = 0$ and $n_I = 0$. However, if $\mu > \mu_c$ then the phase which is usually called the normal quark matter phase is realized in the model. In this phase the quark number density n_q is nonzero, however the isospin density $n_I = 0$ at $\mu_I = 0$. In the particular case with $\alpha = \alpha_0 \approx 0.17$ the behavior of the M-coordinate (gap) of the GMP is presented in Fig. 2, where $\mu_c \approx 0.76M_0$, as a function of μ.

C. Particular case: $\mu = 0$, $\mu_I \neq 0$

In this case the TDP (30) has the following form:

$$\Omega_\nu(M, \Delta) = V_0(M, \Delta) - \frac{\alpha M_0 M}{2\pi} - \int_{-\infty}^{\infty}\frac{dp_1}{2\pi} \times \{E_\Delta^+ + E_\Delta^- - 2\sqrt{p_1^2 + M^2 + \Delta^2}\}. \quad (39)$$

The corresponding system of the gap equations looks like:

$$\frac{2\pi\partial\Omega_\nu(M, \Delta)}{\partial M} \equiv 2M\ln\left(\frac{M^2 + \Delta^2}{M_0^2}\right) - \alpha M_0 - 2M\int_0^\infty dp_1 \times \left\{\frac{E^+}{EE_\Delta^+} + \frac{E^-}{EE_\Delta^-} - \frac{2}{\sqrt{p_1^2 + M^2 + \Delta^2}}\right\} = 0, \quad (40)$$

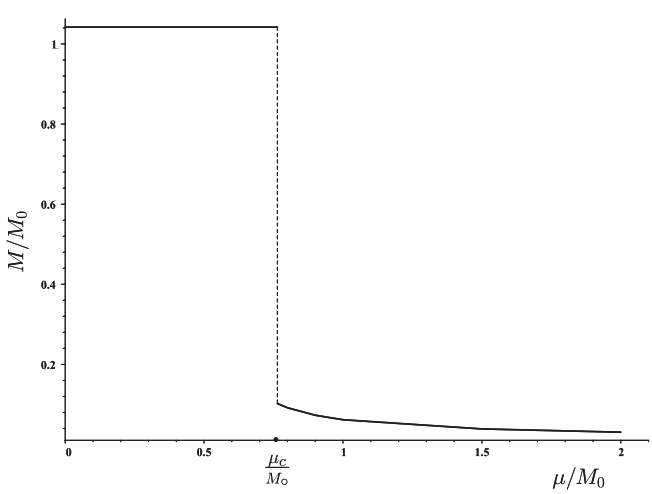

FIG. 2. Dynamical quark mass M vs μ at $\mu_I = 0$ and $\alpha = \alpha_0 \approx 0.17$. Here $\mu_c/M_0 \approx 0.76$.

$$\frac{2\pi\partial\Omega_\nu(M, \Delta)}{\partial\Delta} \equiv 2\Delta\left[\ln\left(\frac{M^2 + \Delta^2}{M_0^2}\right) - \int_0^\infty dp_1 \times \left\{\frac{1}{E_\Delta^+} + \frac{1}{E_\Delta^-} - \frac{2}{\sqrt{p_1^2 + M^2 + \Delta^2}}\right\}\right] = 0, \quad (41)$$

where we have used the notations adopted after formula (19). The coordinates (gap values) $M \equiv M(\nu)$ and $\Delta \equiv \Delta(\nu)$ of the global minimum point of the TDP (39) obey the gap Eqs. (40) and (41). (In the present section we find it convenient to stress explicitly the fact that the GMP is indeed a function of the parameter ν.)

Recall the situation in (3 + 1)-dimensional NJL models with pion condensation, if the bare (current) quark mass is nonzero [13]. In this case at some critical value μ_{Ic} of the isospin chemical potential, which is just the pion meson mass M_π in the vacuum at $\mu = 0$ and $\mu_I = 0$, i.e., $\mu_{Ic} = M_\pi$, there is a continuous second-order phase transition from the vacuum phase (which is realized at $\nu < \nu_c = M_\pi/2$) with $M(\nu) \equiv M(0) \neq 0$, $\Delta(\nu) = 0$ to the pion condensed one (at $\nu > \nu_c$), where $M(\nu) \neq 0$, $\Delta(\nu) \neq 0$. This means that the TDP global minimum point $(M(\nu), \Delta(\nu))$, corresponding to the pion condensed phase, has the following property: $M(\nu) \to M(\nu_c) \equiv M(0)$, $\Delta(\nu) \to 0$, if $\nu \to \nu_{c+}$. Here we again use the notations $\nu = \mu_I/2$ as well as $M(0)$ for the dynamical quark mass in the vacuum.

It turns out that the same qualitative picture of the pion condensed phase transition occurs in the framework of the massive GN model. Indeed, numerical investigations of the TDP (39) show that at some critical point ν_c there is a second-order phase transition from the vacuum phase to the phase with charged pion condensation. It means that the GMP of the TDP (39) is a continuous function vs ν in the critical point $\nu = \nu_c$. Now, in order to define ν_c and to prove that the equality $\nu_c = M_\pi/2$ is also valid in the case of the massive GN model, it is necessary to remark that at $\nu > \nu_c$ the coordinates $(M(\nu), \Delta(\nu))$ of the GMP of the TDP (39) convert the expression in the square brackets of (41) into zero. Moreover, the Eq. (40) is also fulfilled. Since at $\nu = \nu_c$ we have a continuous phase transition, i.e., $\Delta(\nu_c) = 0$, $M(\nu_c) \equiv M(0)$,[2] in the critical point $\nu = \nu_c$ this pair of equations is transformed into the following one

$$\alpha M_0 = 2M(0)\ln\frac{M^2(0)}{M_0^2}, \quad (42)$$

$$\ln\frac{M^2(0)}{M_0^2} = 2\nu_c^2\int_0^\infty dp_1\frac{1}{\sqrt{p_1^2 + M^2(0)}(p_1^2 + M^2(0) - \nu_c^2)}. \quad (43)$$

[2]The quantity $M(0)$ vs α is nothing else than the gap M depicted in Fig. 1 as curve 1.

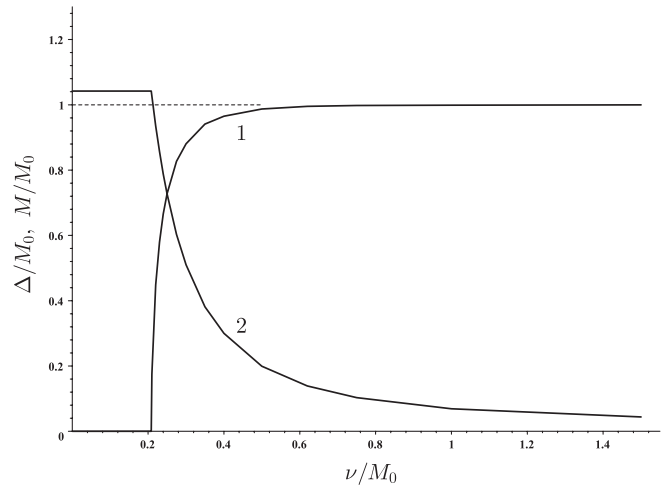

FIG. 3. The gaps Δ (curve 1) and M (curve 2) vs $\nu \equiv \mu_I/2$ in the case $\mu = 0$ and $\alpha = \alpha_0 \approx 0.17$.

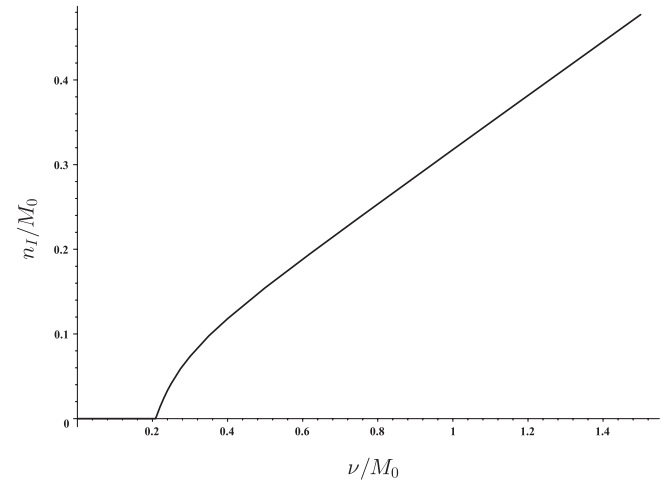

FIG. 4. Isospin density n_I vs ν at $\mu = 0$ and $\alpha = \alpha_0 \approx 0.17$ in the vacuum (at $\nu < \nu_c \approx 0.21M_0$), where $n_I \equiv 0$, and in the PC phase (at $\nu > \nu_c$), where $n_I \neq 0$.

Next, by inserting Eq. (43) into the right-hand side of (42), we find the useful relation

$$\frac{\alpha M_0}{2M(0)} = 2\nu_c^2 \int_0^\infty dp_1 \frac{1}{\sqrt{p_1^2 + M^2(0)}(p_1^2 + M^2(0) - \nu_c^2)}. \tag{44}$$

In the next sections we will study the meson masses in different phases of the model. In particular, we shall there derive an equation for the π-meson mass M_π in the vacuum at $\mu = 0$ and $\mu_I = 0$ (see (57)). Comparing (44) with this equation, it follows that $\nu_c = M_\pi/2$, i.e., the critical value μ_{Ic} is equal to the π-meson mass M_π at $\mu = 0$ and $\mu_I = 0$ for arbitrary values of α. (Of course, one should take into account that the corresponding dynamical quark mass M appearing in this equation is nothing else than the parameter $M(0)$ of the present section.) As a result, the dependence of μ_{Ic} and M_π vs α is presented by the same curve 2 of Fig. 1.

Clearly, at $\nu < \nu_c$ we have a phase which corresponds to the empty space (here both n_q and n_I are equal to zero). Because of this property, we use the notation vacuum for this phase.[3] In the vacuum phase one has $\Delta = 0$, but the gap M is nonzero and does not depend on ν (its behavior vs α is shown in Fig. 1). At $\nu > \nu_c$ the pion condensation (PC) phase with $n_q = 0$ and $n_I \neq 0$ is realized in the model. Inside this phase both gaps M and Δ are nonzero and depend on ν. The isospin $U_{I_3}(1)$ symmetry is spontaneously broken in the PC phase. For the particular parameter value $\alpha = \alpha_0 \approx 0.17$ the behavior of gaps vs ν is shown in Fig. 3, where $\nu_c \approx 0.21M_0$. In Fig. 4 the isospin density n_I vs ν is presented.

D. General case: $\mu \neq 0$, $\mu_I \neq 0$

In this case, starting from the TDP (30) we obtain the following gap equations:

$$\frac{2\pi\partial\Omega_{\mu,\nu}(M,\Delta)}{\partial M} \equiv 2M\ln\left(\frac{M^2+\Delta^2}{M_0^2}\right) - \alpha M_0 - 2M\int_0^\infty dp_1 \times\left\{\frac{E^+}{EE_\Delta^+} + \frac{E^-}{EE_\Delta^-} - \frac{2}{\sqrt{E^2+\Delta^2}} - \frac{E^+\theta(\mu - E_\Delta^+)}{EE_\Delta^+} - \frac{E^-\theta(\mu - E_\Delta^-)}{EE_\Delta^-}\right\} = 0, \tag{45}$$

$$\frac{2\pi\partial\Omega_{\mu,\nu}(M,\Delta)}{\partial \Delta} \equiv 2\Delta\left[\ln\left(\frac{M^2+\Delta^2}{M_0^2}\right) - \int_0^\infty dp_1 \times\left\{\frac{1}{E_\Delta^+} + \frac{1}{E_\Delta^-} - \frac{2}{\sqrt{E^2+\Delta^2}} - \frac{\theta(\mu - E_\Delta^+)}{E_\Delta^+} - \frac{\theta(\mu - E_\Delta^-)}{E_\Delta^-}\right\}\right] = 0. \tag{46}$$

Based on these equations, we have studied the properties of the GMP of the TDP (30) in the particular case of $\alpha = \alpha_0 \approx 0.17$ and found the phase portrait, presented in Fig. 5. There the vacuum, pion condensation as well as three normal quark matter phases I, II, and III are arranged. In the pion condensation phase the gaps Δ and M are nonzero,

[3] By definition, the vacuum is here the phase with zero densities n_q and n_I. However, one should realize that in a most general case the (dynamical) properties of its ground state depend on the values of μ and μ_I. Indeed, in the model under consideration at $\mu_I = 0$ there is an $SU_I(2)$ symmetry of the ground state in the vacuum phase. As a result, all three pions have a common mass. However, at $\mu_I \neq 0$, i.e., when the ground state symmetry is reduced to the $U_{I_3}(1)$ subgroup, π-mesons have different masses in this phase (see Sec. V B 2).

so here the isospin $U_{I_3}(1)$ symmetry is broken spontaneously. Throughout this phase the gaps Δ and M do not depend on μ. It turns out that their dependencies on ν in the PC phase at $\mu \neq 0$ are the same as in the PC phase at $\mu = 0$ (see Fig. 3). For points (ν, μ), taken from the other phases of Fig. 5, the Δ-coordinate of the global minimum point of the TDP is zero (as a result, in these phases the isospin $U_{I_3}(1)$ symmetry remains intact), but the M-coordinate of GMP is not zero. Namely, inside the vacuum phase the gap M does not depend on (ν, μ), i.e., it is a constant. (In particular, here $M \approx 1.04M_0$ at $\alpha = \alpha_0 \approx 0.17$.) Our analysis shows that on the boundary between the vacuum and pion condensation phases the gaps are continuous functions vs μ and ν. Hence, we conclude that a transition from the vacuum to the pion condensed phase or conversely is a second-order one.

It turns out that the gap M is a continuous (μ, ν)-function inside each of the domains I, II, and III of Fig. 5. In contrast, it is changed by a jump when each boundary between the I, II, and III phases is crossed. To become convinced of this, look at Fig. 6, where the behavior of M vs ν at two different fixed values of μ is presented (there, phase II is shrunk to the interval (a_1, b_1) at $\mu = 0.84M_0$ and to the interval (a_2, b_2) at $\mu = 0.94M_0$). As a result, we conclude that on these boundaries there is a first-order phase transition.

Now, let us consider the quark number density n_q (32) as well as the isospin density n_I (33) inside each phase of the model. It is easy to see that for the vacuum phase these quantities are zero, thus justifying the name of these phases. Then, since the gaps Δ and M do not depend on μ inside the pion condensed phase and the relations $E_\Delta^\pm >$

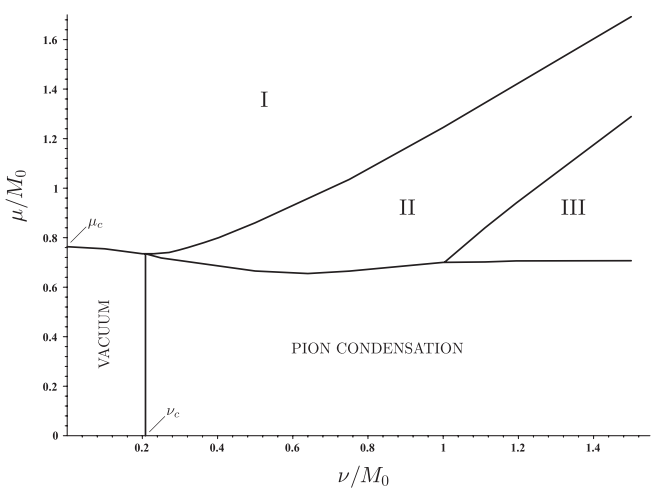

FIG. 5. Phase portrait of the model in terms of μ and $\nu \equiv \mu_I/2$ at $\alpha = \alpha_0 \approx 0.17$. Here $\nu_c/M_0 \approx 0.21$, $\mu_c/M_0 \approx 0.76$. All lines of the figure are curves of first-order phase transitions, except the boundary between the vacuum and PC phase. In the normal quark matter phases I, II, and III pion condensation is absent (their properties are discussed in the text).

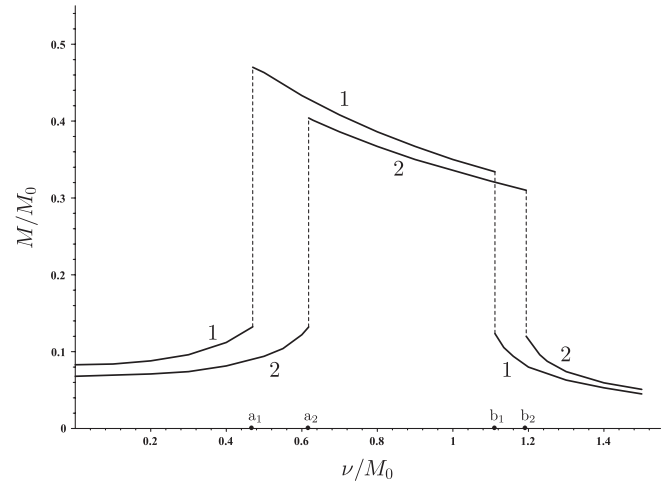

FIG. 6. The behavior of the gap M vs ν in phases I, II, and III at $\mu/M_0 = 0.84$ (curve 1) and $\mu/M_0 = 0.94$ (curve 2) at $\alpha = \alpha_0 \approx 0.17$. Here $a_1 \approx 0.47$, $a_2 \approx 0.62$, $b_1 \approx 1.11$, $b_2 \approx 1.19$.

μ are true here, one can conclude that $n_q \equiv 0$ in this phase and the isospin density n_I vs ν in the PC phase is presented in Fig. 4. For the normal quark matter phases I, II, and III we have $\Delta = 0$, so in order to obtain the expressions for n_q and n_I in these phases one can use the expression (13) of the paper [31] for the quantity $\Omega_{\mu,\nu}(M, \Delta = 0)$. As a result, in phases I, II, and III we have

$$n_q = \frac{\theta(\mu + \nu - M)}{\pi}\sqrt{(\mu + \nu)^2 - M^2} + \frac{\theta(|\mu - \nu| - M)}{\pi}\sqrt{(\mu - \nu)^2 - M^2}\,\mathrm{sign}(\mu - \nu), \tag{47}$$

$$n_I = \frac{\theta(\mu + \nu - M)}{\pi}\sqrt{(\mu + \nu)^2 - M^2} - \frac{\theta(|\mu - \nu| - M)}{\pi}\sqrt{(\mu - \nu)^2 - M^2}\,\mathrm{sign}(\mu - \nu). \tag{48}$$

Then, using the values of the gap M presented in Fig. 6, one can find the corresponding values of densities n_q and n_I shown in the curves of Fig. 7. It is clear from this figure that inside the II-phase $n_q \equiv n_I$. Since $n_q = n_u + n_d$ and $2n_I = n_u - n_d$, where n_u, n_d are the densities of up and down quarks, correspondingly, it is clear from the above-mentioned constraint that in phase II the relation $n_u \equiv -3n_d$ is valid.

Up to now we have studied thermodynamic properties of the model phases. Now the consideration of their dynamical peculiarities is in order. The first point we would like to discuss here is the spectrum of quasiparticles. In condensed matter physics they are simply the one-fermion excitations of the corresponding ground state. Recall, in the most general case the energy spectrum of u-, d-, $\bar{u}$-,

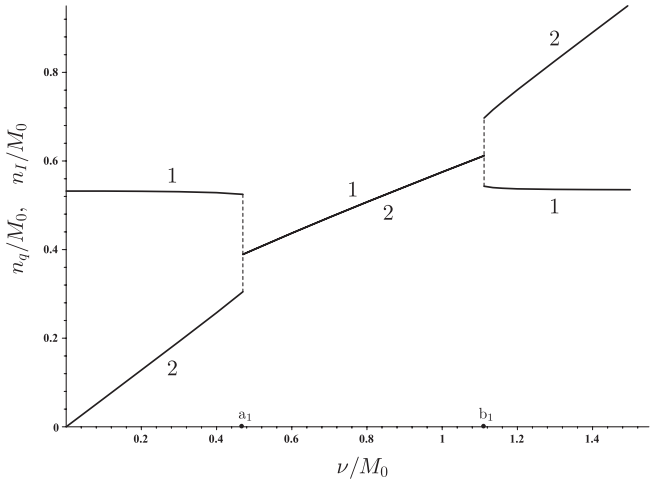

FIG. 7. Quark number density n_q (curve 1) and isospin density n_I (curve 2) vs ν at $\mu/M_0 = 0.84$. Here $a_1 \approx 0.47$, $b_1 \approx 1.11$.

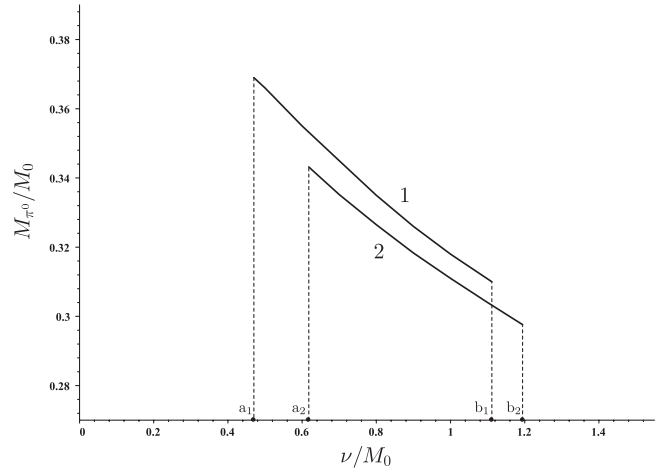

FIG. 8. The behavior of the π^0-meson mass M_{π^0} vs ν in phase II at $\mu/M_0 = 0.84$ (curve 1) and $\mu/M_0 = 0.94$ (curve 2). The values a_i and b_j are given in Fig. 6.

$\bar{d}$-quasiparticles (quarks) are given in (B6) (see Appendix B). It is clear from this formula that in the vacuum phase, where $\Delta = 0$ and $M \approx 1.04M_0$, the energy which is needed for a creation of the u- and d-quasiparticles is always greater than zero. Hence, both u- and d-quarks are the gapped excitations of the vacuum phase. The similar property of a ground state is valid for the PC phase of the model, where also a finite amount of energy is needed to create up and/or down quarks. Because of this reason, the name gapped phases are usually used in these cases.

However, in the case with normal quark matter phases I, II, and III the situation is opposite. Indeed, it is easy to check that in phase I both u- and d-quasiparticles are gapless. It means that there are no energy costs to create these quarks, i.e., there exist space momenta $p_1^\star$ and $p_1^{\star\star}$ such that $p_{0u}(p_1^\star) = 0$ and $p_{0d}(p_1^{\star\star}) = 0$, where $p_{0u}(p_1)$ and $p_{0d}(p_1)$ are the energies given in (B6) of corresponding quasiparticles. (For example, the point ($\nu = 0.2M_0$, $\mu = 0.84M_0$) lies in phase I with $M \approx 0.088M_0$, $\Delta = 0$. Then it is easy to find from (B6) that $p_1^\star \approx 1.04M_0$ and $p_1^{\star\star} \approx 0.63M_0$.) In contrast, in phases II and III only u-quasiparticles are gapless, but d-quarks are gapped. Note, some dynamical effects in dense matter such as transport phenomena (e.g., conductivities etc.) depend essentially on the fact whether or not gapless excitations of the medium are possible. Hence, these effects can occur in a qualitatively different way in phase I on one hand, and in phases II and III, on the other hand.

Finally, it is necessary to remark that the spectrum of mesonic excitations also has a sharp phase dependence. In particular, in Fig. 8 the behavior of the π^0-meson mass in phase II is depicted at two different values of μ. It turns out that in this phase the π^0-meson is a stable particle (at least with respect to strong interactions). However, in the neighboring phases I and III it is no longer a stable particle but a resonance. This fact as well as other peculiarities of the meson spectrum in different phases of the model is the subject of our consideration in the next section.

V. MESON MASSES IN DIFFERENT PHASES

As was noted in Sec. II (see the text after (15)), the effective action (15) can be used for obtaining meson masses in different phases of the model. For this purpose, one should find from the outset all two-point 1PI Green functions (16) of meson fields. These 1PI Green functions are the matrix elements of the 4×4 meson matrix $\Gamma(x - y)$. Then it is necessary to get the Fourier transformation $\bar{\Gamma}(p)$ of the meson matrix $\Gamma(x - y)$ and find its determinant in the rest frame, where the two-component energy-momentum vector p has the form $p = (p_0, 0)$. The equation

$$\det\bar{\Gamma}(p_0) = 0 \tag{49}$$

has in the plane of the variable p_0^2 four (real- or complex-valued) solutions, one of them is the mass squared of the scalar σ-meson, whereas the other three solutions give the mass squared of the pseudoscalar π-mesons.

Detailed investigations of the meson matrix $\bar{\Gamma}(p_0)$ show that its matrix elements of the form $\bar{\Gamma}_{\pi_3 X}(p_0)$ or $\bar{\Gamma}_{X\pi_3}(p_0)$, where $X = \sigma, \pi_1, \pi_2$, are equal to zero in all phases of the model, i.e., the matrix $\bar{\Gamma}(p_0)$ is a reducible one. This means that the neutral pseudoscalar meson, $\pi^0 \equiv \pi_3$, does not mix with the other mesons, $\pi^\pm \equiv (\pi_1 \pm i\pi_2)/\sqrt{2}$ or σ. As a result, one root of the Eq. (49) can be found through the equation $\bar{\Gamma}_{\pi_3\pi_3}(p_0) = 0$, which supplies us with the mass squared of the π^0-meson in different phases of the model. The other three meson masses are the zeros of the determinant of the reduced meson matrix, whose matrix elements are two-point 1PI Green functions of the fields σ, π_1, and π_2.

A. The mass of π^0-meson

The corresponding two-point 1PI Green function looks like:

$$\Gamma_{\pi^0\pi^0}(z) \equiv -\frac{\delta^2 \mathcal{S}^{(2)}_{\rm eff}}{\delta\pi_3(y)\delta\pi_3(x)} = \frac{\delta(z)}{2G} + i\,{\rm Tr}_s\{S_{11}(z)\gamma^5 S_{11}(-z)\gamma^5 + S_{22}(z)\gamma^5 S_{22}(-z)\gamma^5 - S_{12}(z)\gamma^5 S_{21}(-z)\gamma^5 - S_{21}(z)\gamma^5 S_{12}(-z)\gamma^5\}, \quad (50)$$

where $z = x - y$, and the matrix elements $S_{ij}(z)$ of the quark propagator are presented in (B5). Note, the expression (50) is valid for all phases of the model. Now, let us consider it in each phase.

1. Vacuum and normal quark matter phase I: The case $\nu = 0$, $\mu \geq 0$

To illustrate the technique, which was elaborated in details in the framework of NJL models with the color superconductivity phenomenon [30], we start from the most simple case of $\nu = 0$ corresponding to the vacuum and phase I only (see Sec. IV B and Fig. 5). Since for these phases $\Delta = 0$, the last two terms in (50), proportional to $S_{12}(z)$, vanish. The corresponding Fourier transformation of the expression (50) now looks like:

$$\bar{\Gamma}_{\pi^0\pi^0}(p) = \frac{1}{2G} + i\,{\rm Tr}_s \int \frac{d^2k}{(2\pi)^2}\{\bar{S}_{11}(p+k)\gamma^5\bar{S}_{11}(k)\gamma^5 + \bar{S}_{22}(p+k)\gamma^5\bar{S}_{22}(k)\gamma^5\}, \quad (51)$$

where the Fourier transformations $\bar{S}_{ij}(p)$ can be easily determined from (B5). Using in (51) the rest frame system, where $p = (p_0, 0)$, and calculating the trace over spinor indices, we have

$$\bar{\Gamma}_{\pi^0\pi^0}(p_0) = \frac{1}{2G} + 4i\int\frac{d^2k}{(2\pi)^2} \times \frac{E^2 - (k_0+\mu)(p_0+k_0+\mu)}{((k_0+\mu)^2 - E^2)[(k_0+p_0+\mu)^2 - E^2]}, \quad (52)$$

where $E = \sqrt{k_1^2 + M^2}$, and the dynamical quark mass M is given by the value of the M-coordinate of the GMP of the thermodynamic potential. As was noted in Appendix B, in (52) k_0 and $(k_0 + p_0)$ are correspondingly the shorthand notations for $k_0 + i\varepsilon\cdot{\rm sign}(k_0)$ and $(k_0+p_0) + i\varepsilon\cdot{\rm sign}(k_0+p_0)$, where $\varepsilon \to 0_+$. The k_0-integration in (52) is performed along the real axis in the complex k_0-plane. We will close this contour by an infinite arc in the upper half of the complex k_0-plane. Taking into account the above-mentioned rule for the k_0-integration, we have inside the obtained closed contour of the integral in (52) four poles of the integrand which are located in the following points:

$$\begin{aligned}(k_0)_1 &= -E - \mu + i\varepsilon\theta(\mu + E),\\ (k_0)_2 &= E - \mu + i\varepsilon\theta(\mu - E),\\ (k_0)_3 &= -E - \mu - p_0 + i\varepsilon\theta(\mu + E),\\ (k_0)_4 &= E - \mu - p_0 + i\varepsilon\theta(\mu - E).\end{aligned} \quad (53)$$

Since $\mu \geq 0$ and $E \geq 0$, the sum of the corresponding residues of the integrand function in these poles results in the following k_1-integration in (52):

$$\bar{\Gamma}_{\pi^0\pi^0}(p_0) = \frac{1}{2G} + 8\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{E\theta(E-\mu)}{p_0^2 - 4E^2}. \quad (54)$$

To renormalize the expression (54) we use the gap Eq. (22) at $\Delta = \nu = 0$:

$$\frac{1}{2G} = \frac{m}{2M} + 2\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{\theta(E-\mu)}{E}, \quad (55)$$

where we took into account that $m_0/G \equiv m$. Substituting (55) into (54) and using the relation $m \equiv \alpha M_0/\pi$, we have

$$\bar{\Gamma}_{\pi^0\pi^0}(p_0) = \frac{\alpha M_0}{2\pi M} - 2p_0^2\int_0^{\infty}\frac{dk_1}{\pi}\frac{\theta(E-\mu)}{E(4E^2 - p_0^2)}. \quad (56)$$

Note that the quantity $\bar{\Gamma}_{\pi^0\pi^0}(p_0)$ is a multivalued function of the variable p_0^2 which is analytic on some complex Riemann manifold described by several sheets. The expression in the right-hand side of (56) defines $\bar{\Gamma}_{\pi^0\pi^0}(p_0)$ just on the first physical sheet only, which is the whole complex p_0^2-plane, except for the cut $p_0^2 > 4M^2$ along the real axis.

Recall that the mass squared $M^2_{\pi^0}$ of π^0-mesons is the zero of this 1PI Green function vs p_0^2. The zero should lie either on the real axis in the first sheet of the p_0^2-plane (in this case it corresponds to a stable particle with real value of $M^2_{\pi^0}$ such that $0 \leq M^2_{\pi^0} \leq 4M^2$) or in the second sheet, corresponding to a resonance. Since at $\nu = 0$ a mass splitting between π-mesons is absent (see also the remark in footnote [3]), throughout the section we use the notation M_π both for the π^0- as well as for the $\pi^\pm$-meson mass.

It is clear from Fig. 1 (see also Sec. IV B) that in the vacuum phase at $\nu = 0$ the relation $\mu < \mu_c < M$ is valid for arbitrary α-values, so the theta-function in (56) is equal to unity. As a result, we see that in the vacuum the π-meson mass satisfies the following equation:

$$\frac{\alpha M_0}{2M} = 2M_\pi^2\int_0^{\infty}\frac{dk_1}{\sqrt{k_1^2 + M^2}(4k_1^2 + 4M^2 - M_\pi^2)}. \quad (57)$$

Supposing that the quantity M in (57) is just the gap depicted in Fig. 1 as the curve 1, one can solve numerically this equation with respect to the variable M_π. It turns out that the solution M_π lies in the first sheet of the Riemann manifold and hence obeys the relation $M_\pi^2 < 4M^2$. The quantity M_π vs α is shown in Fig. 1 as the curve 2.

In contrast, in the case of the phase I at $\nu = 0$ the corresponding 1PI Green function (56) does not have zeros in the first Riemann sheet of the variable p_0^2, i.e., there are no stable (at least with respect to strong interactions) π-mesonic excitations of the phase I ground state. In this phase all π-mesons are resonances.

2. Vacuum and normal quark matter phases I, II and III: The case $\nu \neq 0$, $\mu \neq 0$

Technically this is a more complicated case, but the main ideas of calculations do not change. So, omitting technical details, one can obtain the following expression for the two-point 1PI Green function of π^0-mesons in the rest frame:

$$\bar{\Gamma}_{\pi^0\pi^0}(p_0) = \frac{\alpha M_0}{2\pi M} - p_0^2 \int_0^\infty \frac{dk_1}{\pi} \frac{1}{E(4E^2 - p_0^2)} \times [\theta(E + \nu - \mu) + \mathrm{sign}(E - \nu)\theta(|E - \nu| - \mu)]. \tag{58}$$

It is also a multivalued function of the variable p_0^2 which is analytical on the same Riemann manifold, where the Green function (56) is defined. On the first Riemann sheet and at real values of p_0^2 such that $0 \leq p_0^2 < 4M^2$ it looks like:

$$\bar{\Gamma}_{\pi^0\pi^0}(p_0) = \frac{\alpha M_0}{2\pi M} - \frac{2p_0}{\sqrt{4M^2 - p_0^2}} \arctan\frac{p_0}{\sqrt{4M^2 - p_0^2}} + \frac{p_0\theta(\mu + \nu - M)}{\sqrt{4M^2 - p_0^2}} \arctan\frac{p_0\sqrt{(\mu + \nu)^2 - M^2}}{(\mu + \nu)\sqrt{4M^2 - p_0^2}} + \frac{p_0\theta(|\mu - \nu| - M)}{\sqrt{4M^2 - p_0^2}} \arctan\frac{p_0\sqrt{(\mu - \nu)^2 - M^2}}{|\mu - \nu|\sqrt{4M^2 - p_0^2}}. \tag{59}$$

Let us, for example, again consider the case $\alpha = \alpha_0 \approx 0.17$, then $M \approx 1.04M_0$ (see Fig. 2). In this case, for the μ and ν values taken from the vacuum phase of Fig. 5, each theta-function in the expression (58) is equal to unity. As a result, the Green function (58) at $\nu \neq 0$ coincides with the π^0 Green function (56) at $\nu = 0$. Hence, in the vacuum phase the mass of the π^0-meson does not depend on both ν and μ. It takes the value $M_{\pi^0} \approx 0.42M_0$ in the case of $\alpha = \alpha_0$. In the general case of arbitrary α-values, the M_{π^0}-mass in the vacuum phase at $\nu \neq 0$ is simply the pion mass at $\nu = 0$ (see the line 2 of Fig. 1).

One can easily check that the expression (59) turns into zero at some point of the interval $0 < p_0 < 2M$ only in the case when (ν, μ) lies in phase II (the corresponding value of p_0 is the mass of the π^0-meson). At some fixed values of μ the behavior of M_{π^0} vs ν is presented in Fig. 8 at $\alpha = \alpha_0$. In contrast, in phases I and III the expression (59) has no zeros in the interval $0 < p_0 < 2M$. Hence, in these phases π^0 is not a stable particle, but rather a resonance.

3. The pion condensation phase

Now, let us study the π^0-mass in the PC phase, where both gaps Δ and M are nonzero. To obtain a compact expression for the two-point 1PI Green function $\bar{\Gamma}_{\pi^0\pi^0}(p_0)$, it is again necessary to eliminate in (50) the coupling constant with the help of the gap Eq. (23), i.e., to use the following relation

$$\frac{1}{2G} = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \left\{ \frac{\theta(E_\Delta^+ - \mu)}{E_\Delta^+} + \frac{\theta(E_\Delta^- - \mu)}{E_\Delta^-} \right\}.$$

Then, after tedious but straightforward calculations which are similar to that of Sec. VA 1, it is possible to find

$$\bar{\Gamma}_{\pi^0\pi^0}(p_0) = (p_0^2 - \mu_I^2) \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \times \frac{E_\Delta^+ + E_\Delta^-}{E_\Delta^- E_\Delta^+ [p_0^2 - (E_\Delta^+ + E_\Delta^-)^2]}. \tag{60}$$

Clearly, the mass of π^0 in the PC phase is equal to the isotopic chemical potential μ_I and does not depend on μ.

B. The masses of σ- and $\pi^\pm$-mesons

As was noted above, to get the masses of σ- and $\pi^\pm$-mesons, it is necessary to find the zeros (in the rest frame with $p = (p_0, 0)$) of the determinant of the reduced meson matrix composed from two-point 1PI Green functions of these particles. Our calculations show that the Green functions are of the form $\bar{\Gamma}_{\sigma\pi_{1,2}}(p_0) \sim \Delta$. So, in the vacuum as well as in phases I, II and III there is no mixing between σ- and $\pi_{1,2}$-fields which leads to a further reduction of the meson matrix. Hence, to find the mass of the σ-meson in these phases, it is sufficient to investigate the separate equation $\bar{\Gamma}_{\sigma\sigma}(p_0) = 0$. The equation $\det\Pi(p_0) = 0$ with

$$\Pi(p_0) \equiv \begin{pmatrix} \bar{\Gamma}_{\pi_1\pi_1}(p_0), & \bar{\Gamma}_{\pi_1\pi_2}(p_0) \\ \bar{\Gamma}_{\pi_2\pi_1}(p_0), & \bar{\Gamma}_{\pi_2\pi_2}(p_0) \end{pmatrix}, \tag{61}$$

then supplies us with the masses of $\pi^\pm$-mesons.

1. σ-meson in vacuum and I, II, III phases

In these phases $\Delta = 0$. So, on the basis of the effective action (15) and using the relation (16) and the methods of Sec. VA 1, it is possible to obtain the most general expression for the two-point 1PI Green function of the σ-meson both in vacuum and in the I, II, III phases of

the model

$$\bar{\Gamma}_{\sigma\sigma}(p_0) = \frac{\alpha M_0}{2\pi M} - (p_0^2 - 4M^2)\int_0^\infty \frac{dk_1}{\pi}\frac{1}{E(4E^2 - p_0^2)} \times [\theta(E + \nu - \mu) + \mathrm{sign}(E - \nu) \times \theta(|E - \nu| - \mu)], \tag{62}$$

where again $E = \sqrt{k_1^2 + M^2}$. Let us now suppose that the pair of chemical potentials (μ, ν) belongs to the vacuum phase of Fig. 5, where, evidently, $M > \mu + \nu$. In this particular case the expression in the square brackets of (62) is equal to 2, so

$$\bar{\Gamma}^{\mathrm{vac}}_{\sigma\sigma}(p_0) = \frac{\alpha M_0}{2\pi M} - 2(p_0^2 - 4M^2)\int_0^\infty \frac{dk_1}{\pi}\frac{1}{E(4E^2 - p_0^2)}. \tag{63}$$

It follows from (63) that in the chiral limit, when $\alpha = 0$ and $M \neq 0$, the σ-meson is a stable particle with mass equal to $2M$. However, at arbitrary small $\alpha > 0$ the zero of the Green function (63), located at the point $p_0^2 = 4M^2$ of the first Riemann sheet at $\alpha = 0$, shifts to the second Riemann sheet, signalling thus that in the vacuum phase of the massive GN model the σ-meson is a resonance. It is quite reasonable that at small values of α the mass of this resonance is near $2M$.

Now remark that for values of μ and ν from the regions I, II or III of Fig. 5 the square brackets of the integrand in (62) cannot be negative. As a result, for all real values of p_0^2 such that $0 < p_0^2 < 4M^2$ the Green function $\bar{\Gamma}_{\sigma\sigma}(p_0)$ is a positive quantity, i.e. it cannot become zero. Thus, in phases I, II and III of the model the σ-meson is also a resonance.[4]

2. $\pi^\pm$-mesons in vacuum and I, II, III phases

The squared masses of $\pi^\pm$-mesons in these phases are given by the zeros of the equation $\det\Pi(p_0) = 0$ in the p_0^2-plane, where $\Pi(p_0)$ is the matrix (61). To find its matrix elements, it is convenient to use in the effective action (15) the new fields $\pi^\pm(x) = (\pi_1(x) \pm i\pi_2(x))/\sqrt{2}$ instead of the old ones, $\pi_{1,2}(x)$. Then, it is natural to define the corresponding Green functions $\Gamma_{\pi^+\pi^-}(x - y)$ etc., where

$$\Gamma_{\pi^+\pi^-}(x - y) = -\frac{\delta^2 \mathcal{S}^{(2)}_{\mathrm{eff}}}{\delta\pi^-(y)\delta\pi^+(x)}$$

etc.[5] The Fourier transformations of these Green functions are connected with the matrix elements of the matrix $\Pi(p_0)$ (61) by the relations

$$\bar{\Gamma}_{\pi_1\pi_1}(p_0) = \bar{\Gamma}_{\pi_2\pi_2}(p_0) = \frac{1}{2}[\bar{\Gamma}_{\pi^+\pi^-}(p_0) + \bar{\Gamma}_{\pi^-\pi^+}(p_0)],$$
$$\bar{\Gamma}_{\pi_1\pi_2}(p_0) = -\bar{\Gamma}_{\pi_2\pi_1}(p_0) = \frac{i}{2}[\bar{\Gamma}_{\pi^+\pi^-}(p_0) - \bar{\Gamma}_{\pi^-\pi^+}(p_0)]. \tag{64}$$

Then, the determinant of the matrix (61) looks like

$$\det\Pi(p_0) = \bar{\Gamma}_{\pi^+\pi^-}(p_0)\cdot\bar{\Gamma}_{\pi^-\pi^+}(p_0). \tag{65}$$

Our straightforward analytical calculations show that

$$\bar{\Gamma}_{\pi^+\pi^-}(p_0) = \bar{\Gamma}_{\pi^0\pi^0}(p_0 + \mu_I),$$
$$\bar{\Gamma}_{\pi^-\pi^+}(p_0) = \bar{\Gamma}_{\pi^0\pi^0}(\mu_I - p_0), \tag{66}$$

where $\bar{\Gamma}_{\pi^0\pi^0}$ is the 1PI Green function of the π^0-meson, presented in (58). Now suppose that at $p_0^2 = M^2_{\pi^0}$ the Green function of the π^0-meson turns into zero, when the chemical potentials (μ, μ_I) are fixed at some values in the vacuum phase or the I, II, III phases. Then, on the basis of the relations (66) it is clear that $\bar{\Gamma}_{\pi^+\pi^-}(p_0) \sim ((p_0 + \mu_I)^2 - M^2_{\pi^0})$ and $\bar{\Gamma}_{\pi^-\pi^+}(p_0) \sim ((\mu_I - p_0)^2 - M^2_{\pi^0})$. As a result, we see that

$$\det\Pi(p_0) \sim [(p_0 + \mu_I)^2 - M^2_{\pi^0}]\cdot[(\mu_I - p_0)^2 - M^2_{\pi^0}] \equiv [p_0^2 - (M_{\pi^0} - \mu_I)^2]\cdot[p_0^2 - (M_{\pi^0} + \mu_I)^2]. \tag{67}$$

Hence, the zeros of the determinant (67), i.e., the quantities $M^2_{\pi^+} = (M_{\pi^0} - \mu_I)^2$ and $M^2_{\pi^-} = (M_{\pi^0} + \mu_I)^2$, can be identified with the mass squared of $\pi^\pm$-mesons.

3. σ- and $\pi^\pm$-mesons in the pion condensation phase

As noted at the beginning of the present section, there arises a mixing between σ and $\pi_{1,2}$ fields in the PC phase of the massive GN model. Thus, to define the mesonic mass spectrum one should find all the zeros of the determinant of the meson matrix, composed of corresponding two-point 1PI Green functions of the form (16). We have found exact analytical expressions for these Green functions and have shown that the determinant has a zero in the point $p_0^2 = 0$. (In order not to overload the paper with rather cumbersome formulas, we do not present here the expressions for these Green functions.) It means that in the PC phase there is a massless bosonic excitation. It can be treated as a Goldstone boson, which is a consequence of the spontaneous breaking of the isospin $U_{I_3}(1)$ symmetry in the PC phase.

It turns out that further information about mesons in the PC phase can be found in the chiral limit, i.e., at $m_0 = 0$.

[4] Strictly speaking, the found resonance character of σ is here associated to the existence of $q\bar{q}$-thresholds enabling the meson decay into a (nonobservable) $q\bar{q}$-pair. Clearly, in order to model e.g. the confinement properties of "more realistic" two-dimensional QCD [32], one should consider a more sophisticated GN model incorporating some suitable prescription for quark confinement (see, e.g., [33]). Within such a model one could then treat the σ-decay into observable pions, $\sigma \to \pi\pi$, which is, however, outside the scope of this paper.

[5] In the phases with zero gap Δ the Green functions of the form $\Gamma_{\pi^+\pi^+}(x - y)$ and $\Gamma_{\pi^-\pi^-}(x - y)$ vanish.

Indeed, in this case the Green functions of the form $\bar{\Gamma}_{\sigma\pi_{1,2}}(p_0)$ are identically equal to zero, so that the σ-meson does not mix with $\pi_{1,2}$-fields. Moreover, it is possible to show that in the massless GN model the Green function $\bar{\Gamma}_{\sigma\sigma}(p_0)$ coincides in the PC phase with the Green function $\bar{\Gamma}_{\pi^0\pi^0}(p_0)$ (see (60)). Because of this relation we conclude that $M_\sigma = M_{\pi^0} = \mu_I$ in the PC phase of the massless GN model.

VI. SUMMARY AND CONCLUSIONS

Recent investigations of the phase diagram of isotopically asymmetric dense quark matter in terms of NJL models show that their pion condensation content is not yet fully understood. Indeed, the number of the charged pion condensation phases of the phase diagram depends strictly on the parameter set of the NJL model. It means that for different values of the coupling constant, cutoff parameter, bare quark mass, etc. just the same NJL model predicts different numbers of pion condensation phases of quark matter both with or without an electric neutrality constraint (see, e.g., [12,14]). Thus, to obtain more objective information about the pion condensation phenomenon of dense quark matter, it is important to invoke alternative approaches. One of them, which qualitatively quite successfully imitates some of the QCD properties (see also the Introduction), is based on the consideration of this phenomenon in the framework of asymptotically free (1 + 1)-dimensional GN models in the leading order of the large N_c-technique.

In the present paper we have studied the phase structure of the massive GN model (1) in terms of quark number (μ)- as well as isospin (μ_I) chemical potentials in the limit $N_c \to \infty$ (for simplicity, the temperature has been taken to be zero). After renormalization (see Sec. III), this model contains two free parameters: M_0, the dynamical quark mass in the vacuum of the corresponding massless GN model, and the renormalization-invariant quark mass $m = m_0/G \equiv \alpha M_0/\pi$ (see also the remark in footnote [1]). In our considerations we often put $\alpha = \alpha_0 \approx 0.17$ in order to have the same relation between the dynamical quark mass M and the π-meson mass M_π in vacuum, i.e., $M/M_\pi = 5/2$, as used in some other NJL model parametrizations [30]. Just at $\alpha = \alpha_0$ the phase portrait of the model is presented in Fig. 5 in terms of μ and $\nu = \mu_I/2$.

First, we have found that at $T = 0$ the charged pion condensation phase of the GN model is realized inside the (noncompact) chemical potential region $\mu_I > M_{\pi^0}$, where μ is not greater than $M_0/\sqrt{2}$ and M_{π^0} is the vacuum mass of the π^0-meson. In this phase the isospin $U_{I_3}(1)$ symmetry is spontaneously broken down and a massless Goldstone bosonic excitation of the ground state appears. Moreover, we have shown that the mass of the π^0-meson in the PC phase is equal to the isospin chemical potential μ_I. All one-quark excitations are found to be gapped particles in this phase. As a result, the quark number density n_q is equal to zero in the PC phase.[6] The same properties of the PC phase are predicted in the framework of some NJL model parametrizations (see, e.g., in [12,14]). In contrast, in the NJL phase diagram the pion condensation phases occupy a compact region and for some parametrization schemes the gapless pion condensation might occur [11–14].

Second, at rather large values of the quark number chemical potential μ we have found a rather rich variety of normal quark matter phases I, II, and III (see Fig. 5), in which the quark number density n_q does not vanish (see Fig. 7). In particular, it turns out that in phase I both u- and d-quarks are gapless quasiparticles. On the contrary, in phases II and III only u-quarks are gapless, whereas d-quarks are gapped. By this reasoning, dynamical effects in transport phenomena for dense quark matter (e.g., conductivities, etc.) can occur in a qualitatively different way in phases I, II, and III. We have studied also the π-meson mass spectrum of these phases and found that in phase I and III the π-mesons are resonances. However, phase II is the so-called "stability island" for π-mesons. Indeed, as it was shown by our numerical calculations, the π^0-meson is a stable excitation of the ground state of this phase. Its mass vs ν is depicted in Fig. 8. The $\pi^\pm$-mesons are also stable in this phase, but their masses are $M_{\pi^\pm} = |M_{\pi^0} \mp \mu_I|$ (see Sec. V B 2). (The same relation between π^0- and $\pi^\pm$-meson masses is also valid inside the vacuum phase of Fig. 5.)

In conclusion, by using the rather simple approach above to the GN phase diagram, we have found a variety of phases with rather rich dynamical contents. A related interesting issue could be the extension of these investigations to inhomogeneous condensates [28]. We hope that our investigation of the phase diagram of the massive GN model will shed some new light on the phase structure of QCD at nonzero baryonic and isotopic densities. Obviously, a more realistic imitation of the QCD phase diagram requires us to include also a nonzero temperature as well as a suitable confinement prescription for quark propagators [33].

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[6] In the gapped phases, PC, or vacuum phases, the relations $E_\Delta^\pm > \mu$ are valid. Then, using (32), it is clear that $n_q \equiv 0$ in these phases.

APPENDIX A. TRACES OF OPERATORS AND THEIR PRODUCTS

Let $\hat{A}, \hat{B}, \ldots$ be some operators in the Hilbert space $\mathbf{H}$ of functions $f(x)$ depending on two real variables, $x \equiv (x^0, x^1)$. In the coordinate representation their matrix elements are $A(x, y), B(x, y), \ldots$, correspondingly, so that

$$(\hat{A}f)(x) \equiv \int d^2y A(x, y) f(y),$$

$$(\hat{A} \cdot \hat{B})(x, y) \equiv \int d^2z A(x, z) B(z, y), \quad \text{etc.}$$

By definition,

$$\mathrm{Tr}\hat{A} \equiv \int d^2x A(x, x),$$

$$\mathrm{Tr}(\hat{A} \cdot \hat{B}) \equiv \int d^2x d^2y A(x, y) B(y, x), \quad \text{etc.} \tag{A1}$$

Each function $f(x) \in \mathbf{H}$ can be considered as an operator $\hat{f}$, acting in this space, with matrix elements $f(x)\delta(x - y)$, where $\delta(x - y)$ is the two-dimensional Dirac delta-function. As a result, one can formally consider the trace of functions, their products as well as the traces of more complicated expressions, such as the products of operators and functions. Indeed, using the definition (A1) we have

$$\begin{aligned}
\mathrm{Tr} f \equiv \mathrm{Tr}\hat{f} &= \delta(0) \int d^2x f(x);\\
\mathrm{Tr}(f_1 f_2) \equiv \mathrm{Tr}(\hat{f}_1 \cdot \hat{f}_2) &= \int d^2x d^2y f_1(x)\delta(x - y) f_2(y)\delta(y - x) = \delta(0) \int d^2x f_1(x) f_2(x);\\
\mathrm{Tr}\{\hat{A}f\} \equiv \mathrm{Tr}\{\hat{A} \cdot \hat{f}\} &= \int d^2x d^2y A(x, y) f(y)\delta(y - x) = \int d^2x A(x, x) f(x);\\
\mathrm{Tr}\{\hat{A}f_1\hat{B}f_2\} \equiv \mathrm{Tr}\{\hat{A} \cdot \hat{f}_1 \cdot \hat{B} \cdot \hat{f}_2\} &= \int d^2x d^2y d^2v d^2u A(x, v) f_1(v)\delta(v - y) B(y, u) f_2(u)\delta(u - x)\\
&= \int d^2u d^2v A(u, v) f_1(v) B(v, u) f_2(u).
\end{aligned} \tag{A2}$$

In particular, it follows from (A2) that

$$\frac{\delta\, \mathrm{Tr}\{\hat{A}f\}}{\delta f(x)} = A(x, x); \qquad \frac{\delta^2\, \mathrm{Tr}\{\hat{A}f_1\hat{B}f_2\}}{\delta f_1(x)\delta f_2(y)} = A(y, x)B(x, y). \tag{A3}$$

Now suppose that $A(x, y) \equiv A(x - y)$, $B(x, y) \equiv B(x - y)$, i.e. that $\hat{A}$, $\hat{B}$ are translationally invariant operators. Then introducing the Fourier transformations of their matrix elements, i.e.,

$$\begin{aligned}
\bar{A}(p) &= \int d^2z A(z) e^{ipz},\\
A(z) &= \int \frac{d^2p}{(2\pi)^2} \bar{A}(p) e^{-ipz}, \quad \text{etc.},
\end{aligned} \tag{A4}$$

where $z = x - y$, it is possible to obtain from the above formulas

$$\mathrm{Tr}\,\hat{A} = A(0) \int d^2x = \int \frac{d^2p}{(2\pi)^2} \bar{A}(p) \int d^2x. \tag{A5}$$

If there is an operator function $F(\hat{A})$, where $\hat{A}$ is a translationally invariant operator, then in the coordinate representation its matrix elements depend on the difference $(x - y)$. Obviously, it is possible to define the Fourier transformations $\overline{F(A)}(p)$ of its matrix elements, and the following relations are valid ($\bar{A}(p)$ is the Fourier transformation for the matrix element $A(x - y)$):

$$\begin{aligned}
\overline{F(A)}(p) &= F(\bar{A}(p));\\
\mathrm{Tr}F(\hat{A}) &= \int \frac{d^2p}{(2\pi)^2} F(\bar{A}(p)) \int d^2x.
\end{aligned} \tag{A6}$$

Finally, suppose that $\hat{A}$ is an operator in some internal n-dimensional vector space, in addition. Evidently, the same is valid for the Fourier transformation $\bar{A}(p)$ which is now some $n \times n$ matrix. Let $\lambda_i(p)$ be eigenvalues of the $n \times n$ matrix $\bar{A}(p)$, where $i = 1, 2, \ldots, n$. Then

$$\begin{aligned}
\mathrm{Tr}\, F(\hat{A}) &= \int \frac{d^2p}{(2\pi)^2} \mathrm{tr}F(\bar{A}(p)) \int d^2x\\
&= \sum_{i=1}^{n} \int \frac{d^2p}{(2\pi)^2} F(\lambda_i(p)) \int d^2x.
\end{aligned} \tag{A7}$$

In this formula we use the notation "tr" for the trace of any operator in the internal n-dimensional vector space only, whereas the symbol "Tr" means the trace of an operator both in the coordinate and internal spaces.

APPENDIX B. QUARK PROPAGATOR

It is clear from (12) that the quark propagator S_0 is the following 2×2 matrix in the two-dimensional flavor space:

$$S_0 \equiv \begin{pmatrix} S_{11}, & S_{12} \\ S_{21}, & S_{22} \end{pmatrix} = \begin{pmatrix} D_+, & D_{12} \\ D_{21}, & D_- \end{pmatrix}^{-1}, \tag{B1}$$

where (the summation over $\alpha = 0, 1$ is implied)

$$D_\pm = i\gamma^\alpha\partial_\alpha - M + (\mu \pm \nu)\gamma^0,$$
$$D_{12} = D_{21} = -i\gamma^5\Delta. \tag{B2}$$

The connection between S_{ij} and D_{ij} is the following:

$$S_{11} = [D_+ - D_{12}D_-^{-1}D_{21}]^{-1}, \qquad S_{21} = -D_-^{-1}D_{21}S_{11},$$
$$S_{22} = [D_- - D_{21}D_+^{-1}D_{12}]^{-1}, \qquad S_{12} = -D_+^{-1}D_{12}S_{22}. \tag{B3}$$

It is easy to establish the following relations:

$$D_\pm = \int \frac{d^2p}{(2\pi)^2} e^{-ip(x-y)}\{(p_0 + \mu - E^\mp)\gamma^0\Lambda_+ + (p_0 + \mu + E^\pm)\gamma^0\Lambda_-\},$$
$$(D_\pm)^{-1} = \int \frac{d^2p}{(2\pi)^2} e^{-ip(x-y)} \times \left\{\frac{\Lambda_+\gamma^0}{p_0 + \mu - E^\mp} + \frac{\Lambda_-\gamma^0}{p_0 + \mu + E^\pm}\right\}, \tag{B4}$$

where $E^\pm = E \pm \nu$, $E = \sqrt{p_1^2 + M^2}$, $\Lambda_\pm = \frac{1}{2} \times (1 \pm \frac{\gamma^0(\gamma_1 p_1 + M)}{E})$, and $\gamma_1 = -\gamma^1$.[7] Using the relations (B3), it is possible to obtain from (B2) and (B4):

$$S_{11} = \int \frac{d^2p}{(2\pi)^2} e^{-ip(x-y)}\left\{\frac{p_0 + \mu + E^-}{(p_0 + \mu)^2 - (E_\Delta^-)^2}\gamma^0\bar{\Lambda}_- + \frac{p_0 + \mu - E^+}{(p_0 + \mu)^2 - (E_\Delta^+)^2}\gamma^0\bar{\Lambda}_+\right\},$$
$$S_{22} = \int \frac{d^2p}{(2\pi)^2} e^{-ip(x-y)}\left\{\frac{p_0 + \mu + E^+}{(p_0 + \mu)^2 - (E_\Delta^+)^2}\gamma^0\bar{\Lambda}_- + \frac{p_0 + \mu - E^-}{(p_0 + \mu)^2 - (E_\Delta^-)^2}\gamma^0\bar{\Lambda}_+\right\},$$
$$S_{12} = -i\Delta \int \frac{d^2p}{(2\pi)^2} e^{-ip(x-y)}\left\{\frac{\gamma^5\bar{\Lambda}_+}{(p_0 + \mu)^2 - (E_\Delta^-)^2} + \frac{\gamma^5\bar{\Lambda}_-}{(p_0 + \mu)^2 - (E_\Delta^+)^2}\right\},$$
$$S_{21} = -i\Delta \int \frac{d^2p}{(2\pi)^2} e^{-ip(x-y)}\left\{\frac{\gamma^5\bar{\Lambda}_+}{(p_0 + \mu)^2 - (E_\Delta^+)^2} + \frac{\gamma^5\bar{\Lambda}_-}{(p_0 + \mu)^2 - (E_\Delta^-)^2}\right\}, \tag{B5}$$

where $\bar{\Lambda}_\pm = \frac{1}{2}(1 \pm \frac{\gamma^0(\gamma_1 p_1 - M)}{E})$ and p_0 in the integrand is a shorthand notation for $p_0 + i\varepsilon \cdot \mathrm{sign}(p_0)$, where $\varepsilon \to 0_+$. This prescription for the quantity p_0 correctly implements the role of the quantities μ and μ_I as the chemical potentials and preserves the causality of the theory [34]. It is worth also to note the following useful relations:

$$\gamma^5\bar{\Lambda}_\pm\gamma^5 = \Lambda_\pm, \qquad \gamma^0\bar{\Lambda}_\pm\gamma^0 = \Lambda_\mp.$$

The poles of the matrix elements (B5) of the quark propagator in the energy-momentum space give the dispersion lows for quasiparticles, i.e., the momentum dependence of the quark (p_{0u}, p_{0d}) and antiquark ($p_{0\bar{u}}$, $p_{0\bar{d}}$) energies, in a medium

$$p_{0u} = E_\Delta^- - \mu, \qquad p_{0d} = E_\Delta^+ - \mu,$$
$$p_{0\bar{u}} = -(E_\Delta^+ + \mu), \qquad p_{0\bar{d}} = -(E_\Delta^- + \mu). \tag{B6}$$

Strictly speaking, the quantities p_{0u}, p_{0d} from (B6) are the energies necessary for the creation of quarks with momentum p_1, whereas $p_{0\bar{u}}$, $p_{0\bar{d}}$ is the energy necessary for the annihilation of antiquarks.

[7] In (1 + 1)-dimensions the gamma matrices have the form:
$$\gamma^0 = \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix}.$$

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