Finite size effects in the Gross-Neveu model with isospin chemical potential

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The properties of the two-flavored Gross-Neveu model in the (1 + 1)-dimensional $R^1 \times S^1$ spacetime with compactified space coordinate are investigated in the presence of the isospin chemical potential μ_I. The consideration is performed in the limit $N_c \to \infty$, i.e. in the case with infinite number of colored quarks. It is shown that at $L = \infty$ (L is the length of the circumference S^1) the pion condensation phase is realized for arbitrary small nonzero μ_I. At finite values of L, the phase portraits of the model in terms of parameters $\nu \sim \mu_I$ and $\lambda \sim 1/L$ are obtained both for periodic and antiperiodic boundary conditions of the quark field. It turns out that in the plane (λ, ν) there is a strip $0 \leq \lambda < \lambda_c$ which lies as a whole inside the pion condensed phase. In this phase the pion condensation gap is an oscillating function vs both λ (at fixed ν) and ν (at fixed λ).

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I. INTRODUCTION

It is well known that QCD is a fundamental theory of strong interactions both in the vacuum and in hot and/or dense baryonic matter. However, it can be successfully used only in the region of high energies, temperatures, and densities (or chemical potentials), where a weak-coupling expansion is applicable. Away from this region, different nonperturbative methods or effective theories such as chiral effective Lagrangians as well as Nambu–Jona-Lasinio type models (see, e.g., the papers [1–4] and references therein), are usually employed for the consideration of light meson physics, phase transitions in dense quark matter, etc. In particular, motivated by the fact that in heavy-ion collisions and compact stars the hadronic matter is isotopically asymmetric, different QCD-like effective models were studied at nonzero isospin chemical potential μ_I [5–9]. There the charged pion condensation phenomenon, which is generated if μ_I is greater than the pion mass m_π, was also considered.

In all the above-mentioned papers the effective models are (i) field theories in usual (3 + 1)-dimensional spacetime, and (ii) they are employed for the description of QCD at *rather low* energies and densities. At the same time there is another class of theories that can be used as a laboratory for a qualitative consideration of QCD at *arbitrary* energies. These are the so-called Gross-Neveu (GN) type models, i.e. two-dimensional quantum field theories with four-fermion interactions [10–12]. Renormalizability, asymptotic freedom, as well as the spontaneous breaking of chiral symmetry (in the vacuum) are the most fundamental features that are inherent both for QCD and all GN type models. In addition, the GN phase portrait in terms of baryon chemical potential μ_B vs temperature resembles qualitatively to a great extent the QCD phase diagram [13–17]. Because of their relative simplicity in the leading order of a large-N_c expansion (N_c is a number of colored quarks), it is very convenient to use GN models for considering such a phenomenon of dense QCD as color superconductivity [16,18] and to elaborate new nonperturbative methods of quantum field theory [19–21]. Moreover, the influence of the space compactification on chiral symmetry breaking both in the vacuum ($\mu_B = 0$) [22] and in dense baryon matter ($\mu_B \neq 0$) [23] was studied in terms of GN models (see also the appropriate papers [24–26]).

Before investigating different physical effects relevant to a real (3 + 1)-dimensional world in the framework of two-dimensional GN models, let us recall that there is a no-go theorem forbidding the spontaneous breaking of continuous symmetries in two dimensions [27]. However, at the present time it is well understood (see, e.g., the discussion in [11,15–17]) that in the limit $N_c \to \infty$ this no-go theorem does not apply. This makes it possible to study symmetry breaking effects in terms of GN models as well, but only in the leading order of the $1/N_c$ expansion, where most low dimensional theories are exactly solvable. In this sense, for $N_c \to \infty$, low dimensional quark models are physically more tractable and appealing than at finite N_c.

In the present paper the pion condensation phenomenon is investigated in the framework of the two-dimensional GN model with two massless quark flavors. In particular, we shall study the influence of the finiteness of the system size on this phenomenon. So our consideration is performed in a spacetime with nontrivial topology, i.e. on the $R^1 \times S^1$ manifold with compactified space coordinate, and the GN model is extended by an isospin chemical potential μ_I (for simplicity, we put $\mu_B = 0$). Obviously, the latter issue is motivated by the physics of compact stars, where pion condensation might be realized as a consequence of the isotopic asymmetry of baryon matter.

Since all the calculations are carried out on the basis of the leading order of $1/N_c$ expansion (i.e. in the case $N_c \to \infty$) we expect that all conclusions concerning the pion condensation phenomenon, caused by a spontaneous breaking of the continuous isospin symmetry, remain qualitatively valid for real QCD.

II. THE CASE OF $R^1 \times R^1$ SPACETIME

A. The model and its thermodynamic potential

We consider a two-dimensional model which describes dense quark matter with two massless quark flavors (u and d quarks). Its Lagrangian has the form

$$L = \bar{q}\Big[\gamma^\nu \mathrm{i}\partial_\nu + \frac{\mu_B}{3}\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0\Big]q + \frac{G}{N_c}[(\bar{q}q)^2 + (\bar{q}\mathrm{i}\gamma^5\vec{\tau}q)^2], \tag{1}$$

where the quark field $q(x) \equiv q_{i\alpha}(x)$ is a flavor doublet ($i = 1, 2$ or $i = u, d$) and color N_c-plet ($\alpha = 1, \ldots, N_c$) as well as a two-component Dirac spinor (the summation in (1) over flavor, color, and spinor indices is implied); τ_k ($k = 1, 2, 3$) are Pauli matrices; the baryon chemical potential μ_B in (1) is responsible for the nonzero baryon density of quark matter, whereas the isospin chemical potential μ_I is switched on in order to study properties of quark matter at nonzero isospin densities (in this case the densities of u and d quarks are different). Evidently, the model (1) is a generalization of the two-dimensional Gross-Neveu model [10] with a single massless quark color N_c-plet to the case of two quark flavors and additional chemical potentials. As a result, we have in the case under consideration a more complicated chiral symmetry group. Indeed, at $\mu_I = 0$ apart from the global color $SU(N_c)$ symmetry, the Lagrangian (1) is invariant under transformations from the chiral $SU_L(2) \times SU_R(2)$ group. However, at $\mu_I \neq 0$ this symmetry is reduced to $U_{I_3L}(1) \times U_{I_3R}(1)$, where $I_3 = \tau_3/2$ is the third component of the isospin operator (here and above the subscripts L, R mean that the corresponding group acts only on the left, right-handed spinors, respectively). Evidently, this symmetry can also be presented as $U_{I_3}(1) \times U_{AI_3}(1)$, where $U_{I_3}(1)$ is the isospin subgroup and $U_{AI_3}(1)$ is the axial isospin subgroup. Quarks are transformed under these subgroups as $q \to \exp(\mathrm{i}\alpha\tau_3)q$ and $q \to \exp(\mathrm{i}\alpha\gamma^5\tau_3)q$, respectively.[1]

The linearized version of the Lagrangian (1), which contains composite bosonic fields $\sigma(x)$ and $\pi_a(x)$ ($a = 1, 2, 3$), has the following form (in what follows, we use the notation $\mu \equiv \mu_B/3$ for the quark chemical potential):

$$\tilde{L} = \bar{q}\Big[\gamma^\nu \mathrm{i}\partial_\nu + \mu\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0 - \sigma - \mathrm{i}\gamma^5\pi_a\tau_a\Big]q - \frac{N_c}{4G}[\sigma\sigma + \pi_a\pi_a]. \tag{2}$$

From the Lagrangian (2) one gets the equations for the bosonic fields

$$\sigma(x) = -2\frac{G}{N_c}(\bar{q}q); \qquad \pi_a(x) = -2\frac{G}{N_c}(\bar{q}\mathrm{i}\gamma^5\tau_a q). \tag{3}$$

Obviously, the Lagrangian (2) is equivalent to the Lagrangian (1) when using Eqs. (3). Furthermore, it is clear from (3) and footnote [1] that the bosonic fields transform under the isospin $U_{I_3}(1)$ and axial isospin $U_{AI_3}(1)$ subgroups in the following manner:

$$\begin{aligned}
&U_{I_3}(1)\colon\ \sigma \to \sigma; \qquad \pi_3 \to \pi_3;\\
&\pi_1 \to \cos(2\alpha)\pi_1 + \sin(2\alpha)\pi_2;\\
&\pi_2 \to \cos(2\alpha)\pi_2 - \sin(2\alpha)\pi_1,\\
&U_{AI_3}(1)\colon\ \pi_1 \to \pi_1; \qquad \pi_2 \to \pi_2;\\
&\sigma \to \cos(2\alpha)\sigma + \sin(2\alpha)\pi_3;\\
&\pi_3 \to \cos(2\alpha)\pi_3 - \sin(2\alpha)\sigma.
\end{aligned} \tag{4}$$

Starting from the theory (2), one obtains in the leading order of the large-N_c expansion (i.e. in the one-fermion loop approximation) the following path integral expression for the effective action $\mathcal{S}_{\mathrm{eff}}(\sigma, \pi_a)$ of the bosonic $\sigma(x)$ and $\pi_a(x)$ fields:

$$\exp(\mathrm{i}\mathcal{S}_{\mathrm{eff}}(\sigma, \pi_a)) = N' \int [d\bar{q}][dq]\exp\Big(\mathrm{i}\int \tilde{L}d^2x\Big),$$

where

$$\mathcal{S}_{\mathrm{eff}}(\sigma, \pi_a) = -N_c\int d^2x\Big[\frac{\sigma^2 + \pi_a^2}{4G}\Big] + \tilde{\mathcal{S}}_{\mathrm{eff}}, \tag{5}$$

N' is a normalization constant. The quark contribution to the effective action, i.e. the term $\tilde{\mathcal{S}}_{\mathrm{eff}}$ in (5), is given by

$$\exp(\mathrm{i}\tilde{\mathcal{S}}_{\mathrm{eff}}) = N' \int [d\bar{q}][dq]\exp\Big(\mathrm{i}\int [\bar{q}\mathrm{D}q]d^2x\Big) = [\mathrm{Det}D]^{N_c}. \tag{6}$$

In (6) we have used the notation $\mathrm{D} \equiv D \times \mathrm{I}_c$, where I_c is the unit operator in the N_c-dimensional color space and

$$D \equiv \gamma^\nu \mathrm{i}\partial_\nu + \mu\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0 - \sigma - \mathrm{i}\gamma^5\pi_a\tau_a \tag{7}$$

is the Dirac operator, which acts in the flavor, spinor, as well as coordinate spaces only. Using the general formula $\mathrm{Det}D = \exp\mathrm{Tr}\ln D$, one obtains for the effective action the following expression:

$$\mathcal{S}_{\mathrm{eff}}(\sigma, \pi_a) = -N_c\int d^2x\Big[\frac{\sigma^2 + \pi_a^2}{4G}\Big] - \mathrm{i}N_c\mathrm{Tr}_{sfx}\ln D, \tag{8}$$

[1] Recall for the following that $\exp(\mathrm{i}\alpha\tau_3) = \cos\alpha + \mathrm{i}\tau_3\sin\alpha$, $\exp(\mathrm{i}\alpha\gamma^5\tau_3) = \cos\alpha + \mathrm{i}\gamma^5\tau_3\sin\alpha$.

where the Tr-operation stands for the trace in spinor (s), flavor (f), as well as two-dimensional coordinate (x) spaces, respectively. Using (8), we obtain the thermodynamic potential (TDP) $\Omega_{\mu,\mu_I}(\sigma,\pi_a)$ of the system:

$$\Omega_{\mu,\mu_I}(\sigma,\pi_a) \equiv -\frac{\mathcal{S}_{\rm eff}(\sigma,\pi_a)}{N_c\int d^2x}\bigg|_{\sigma,\pi_a={\rm const}} = \frac{\sigma^2+\pi_a^2}{4G} + {\rm i}\frac{{\rm Tr}_{sfx}\ln D}{\int d^2x}$$
$$= \frac{\sigma^2+\pi_a^2}{4G} + {\rm i}\,{\rm Tr}_{sf}\int\frac{d^2p}{(2\pi)^2}\ln\Big(\not p + \mu\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0 - \sigma - {\rm i}\gamma^5\pi_a\tau_a\Big), \tag{9}$$

where the σ and π_a fields are now x-independent quantities, and in the round brackets of (9) just the momentum space representation, $\bar{D}$, of the Dirac operator D appears. Evidently, ${\rm Tr}_{sf}\ln\bar{D} = \sum_i \ln\epsilon_i$, where the summation over all four eigenvalues ϵ_i of the 4×4 matrix $\bar{D}$ is implied and

$$\epsilon_{1,2,3,4} = -\sigma \pm \sqrt{(p_0+\mu)^2 - p_1^2 - \pi_a^2 + (\mu_I/2)^2 \pm \mu_I\sqrt{(p_0+\mu)^2 - \pi_1^2 - \pi_2^2}}. \tag{10}$$

Hence,

$$\Omega_{\mu,\mu_I}(\sigma,\pi_a) = \frac{\sigma^2+\pi_a^2}{4G} + {\rm i}\int\frac{d^2p}{(2\pi)^2}\ln(\epsilon_1\epsilon_2\epsilon_3\epsilon_4)$$
$$= \frac{\sigma^2+\pi_a^2}{4G} + {\rm i}\int\frac{d^2p}{(2\pi)^2}\ln\{[(p_0+\mu)^2-\varepsilon_+^2]\times[(p_0+\mu)^2-\varepsilon_-^2]\}, \tag{11}$$

where

$$\varepsilon_\pm = \sqrt{\Big(\sqrt{p_1^2+\sigma^2+\pi_3^2} \pm \frac{\mu_I}{2}\Big)^2 + \pi_1^2 + \pi_2^2}. \tag{12}$$

The TDP $\Omega_{\mu,\mu_I}(\sigma,\pi_a)$ is symmetric under the transformations $\mu\to-\mu$ and/or $\mu_I\to-\mu_I$. Hence, it is sufficient to consider only the region $\mu\ge 0$, $\mu_I\ge 0$. In this case, one can integrate in (11) over p_0 with the help of the formula

$$\int\frac{dp_0}{2\pi}\ln[(p_0+a)^2-b^2] = \frac{\rm i}{2}\{|a-b|+|a+b|\} \tag{13}$$

(which is valid up to an infinite constant independent of quantities a, b) and obtain

$$\Omega_{\mu,\mu_I}(\sigma,\pi_a) = \frac{\sigma^2+\pi_a^2}{4G} - \int_{-\infty}^{\infty}\frac{dp_1}{4\pi}\{|\varepsilon_+-\mu| + |\varepsilon_++\mu| + |\varepsilon_--\mu| + |\varepsilon_-+\mu|\}$$
$$= \frac{\sigma^2+\pi_a^2}{4G} - \int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\{\varepsilon_+ + \varepsilon_- + (\mu-\varepsilon_+)\theta(\mu-\varepsilon_+) + (\mu-\varepsilon_-)\theta(\mu-\varepsilon_-)\}. \tag{14}$$

(To get the second line in (14) we used the relations $|\varepsilon_\pm+\mu| = \varepsilon_\pm+\mu$ and $\theta(x)+\theta(-x)=1$.) In what follows we are going to investigate the μ, μ_I-dependence of the global minimum point of the function $\Omega_{\mu,\mu_I}(\sigma,\pi_a)$ vs σ, π_a. To simplify the task, let us note that both the quasiparticle energies (12) and hence the TDP (14) depend effectively only on the two combinations $\sigma^2+\pi_3^2$ and $\pi_1^2+\pi_2^2$ of the bosonic fields, which are invariants with respect to the $U_{I_3}(1)\times U_{AI_3}(1)$ group, as is easily seen from (4). In this case, without loss of generality, one can put $\pi_2=\pi_3=0$ in (14), and study the TDP as a function of only two variables, $M\equiv\sigma$ and $\Delta\equiv\pi_1$. Then the global minimum point of the TDP $\Omega_{\mu,\mu_I}(M,\Delta)$,

$$\Omega_{\mu,\mu_I}(M,\Delta) = \frac{M^2+\Delta^2}{4G} - \int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\{E_\Delta^+ + E_\Delta^- + (\mu-E_\Delta^+)\theta(\mu-E_\Delta^+) + (\mu-E_\Delta^-)\theta(\mu-E_\Delta^-)\}, \tag{15}$$

is the solution of the system of gap equations

$$0 = \frac{\partial\Omega_{\mu,\mu_I}(M,\Delta)}{\partial M} \equiv \frac{M}{2G} - M\int_{-\infty}^{\infty}\frac{dp_1}{2\pi E}\Big\{\frac{\theta(E_\Delta^+-\mu)E^+}{E_\Delta^+} + \frac{\theta(E_\Delta^--\mu)E^-}{E_\Delta^-}\Big\},$$
$$0 = \frac{\partial\Omega_{\mu,\mu_I}(M,\Delta)}{\partial\Delta} \equiv \frac{\Delta}{2G} - \Delta\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\Big\{\frac{\theta(E_\Delta^+-\mu)}{E_\Delta^+} + \frac{\theta(E_\Delta^--\mu)}{E_\Delta^-}\Big\}, \tag{16}$$

where $E_\Delta^\pm = \sqrt{(E^\pm)^2+\Delta^2}$, $E^\pm = E\pm\frac{\mu_I}{2}$, and $E=\sqrt{p_1^2+M^2}$. Evidently, the coordinates M and Δ of the global minimum point of the TDP (15) supply us with two order parameters (gaps), which are proportional to the ground state expectation values of the form $\langle\bar{q}q\rangle$ and $\langle\bar{q}{\rm i}\gamma^5\tau_1 q\rangle$, respectively. If the gap M is nonzero, then in the ground state of the model the axial isospin symmetry $U_{AI_3}(1)$ (at $\mu_I\ne 0$) is spontaneously broken down. Moreover, if the gap $\Delta\ne 0$, then in the ground state, corresponding to the phase with charged pion condensation, the isospin $U_{I_3}(1)$ symmetry is spontaneously broken down.

B. Pion condensation: the case of $\mu = 0$, $\mu_I \neq 0$

Since at $\mu_I = 0$, $\mu\ne 0$ the phase structure of different GN models was reasonably well studied both in two dimensions [13,15–17] and in three dimensions [28] (in the last case the four-fermion theories are also renormalizable), in this subsection we shall study for simplicity the model (1) only at zero quark chemical potential, i.e. at $\mu = 0$, but $\mu_I \neq 0$. The corresponding TDP will be denoted as $\Omega_{\mu_I}(M, \Delta)$ and can be obtained from (15)

$$\Omega_{\mu_I}(M, \Delta) = \frac{M^2 + \Delta^2}{4G} - \int_{-\infty}^{\infty} \frac{dp_1}{2\pi}\{E_\Delta^+ + E_\Delta^-\}$$
$$\equiv V_0(\rho) - \int_{-\infty}^{\infty} \frac{dp_1}{2\pi}\{E_\Delta^+ + E_\Delta^- - 2\sqrt{\rho^2 + p_1^2}\}, \tag{17}$$

where $\rho = \sqrt{M^2 + \Delta^2}$ and $V_0(\rho)$ is the TDP of the system in the vacuum, i.e. at $\mu_I = 0$. In the vacuum the TDP is usually called effective potential:

$$V_0(\rho) = \frac{\rho^2}{4G} - 2\int_{-\infty}^{\infty} \frac{dp_1}{2\pi}\sqrt{\rho^2 + p_1^2}. \tag{18}$$

It is easily seen that both the TDP (17) and the effective potential (18) are formally ultraviolet (UV) divergent quantities. So, a few words are needed about the renormalization procedure of the initial model. It is well known that all four-fermion theories of the type (1) are renormalizable in two-dimensional spacetime [10]. Moreover, in the leading order of the large-N_c expansion only the coupling constant should be renormalized in order to obtain finite (renormalized) expressions for different quantities (see, e.g., [14]). It means that the bare coupling constant G of the model (1) depends on the cutoff parameter Λ, $G \equiv G(\Lambda)$, in such a way that all UV divergences, arising from loop integrations when $\Lambda \to \infty$, are compensated by corresponding terms of $G(\Lambda)$. As a consequence, in the limit $\Lambda \to \infty$ one must necessarily obtain finite expressions for physical quantities. Of course, different renormalization procedures result in different expressions for the bare coupling constant $G(\Lambda)$. However, physical consequences of the theory do not depend on the concrete renormalization scheme. Taking this last remark into account, let us next discuss how to obtain a finite renormalized expression for the TDP (17). We see here two ways. On the one hand, one could find an expression for the bare coupling constant G such that the UV divergence, arising from the integral in the first line of (17), would be compensated by the term with G. Evidently, in this case G depends both on the cutoff Λ and μ_I. However, we find it more convenient to consider the second way. In this case, one should first of all note that the integral in the second line of (17) is a convergent quantity, and the whole UV divergence is located in the effective potential $V_0(\rho)$. Hence, there is a possibility to remove UV divergences using a bare coupling constant which does not depend on μ_I. Namely, let us choose

$$\frac{1}{2G} = \frac{2}{\pi}\int_0^\Lambda dp_1 \frac{1}{\sqrt{M_0^2 + p_1^2}} = \frac{2}{\pi}\ln\left(\frac{\Lambda + \sqrt{M_0^2 + \Lambda^2}}{M_0}\right), \tag{19}$$

where M_0 is the dynamical mass of quarks in the vacuum (for more details, see the appendix). Then, substituting (19) into (18) and restricting there the range of integration by using the cutoff parameter Λ, it is possible to obtain for $\Lambda \gg M_0$ the expression (moreover, we omit an inessential infinite constant independent of ρ):

$$V_0(\rho) = \frac{\rho^2}{2\pi}\left[\ln\left(\frac{\rho^2}{M_0^2}\right) - 1\right]. \tag{20}$$

Since M_0 might be considered as a free model parameter, it follows from (19) and (20) that the renormalization procedure of the GN model is accompanied by the dimensional transmutation phenomenon. Indeed, in the initial unrenormalized expressions both for $\Omega_{\mu_I}(M, \Delta)$ and $V_0(\rho)$ [see (17) and (18), respectively] the dimensionless coupling constant G is present, whereas after renormalization the effective potential (20) is characterized by a dimensional free model parameter M_0.

Because of the relation (20), one can show that the gap equations for the renormalized TDP (17) might have no more than three different solutions. Two of them, ($M = 0$, $\Delta = 0$) and ($M = 0$, $\Delta = M_0$), are present at arbitrary values of $\mu_I \geq 0$, whereas the third one, ($M = M_0$, $\Delta = 0$), appears only at $\mu_I < M_0\sqrt{2}$. However, for arbitrary $\mu_I > 0$ a global minimum point of the TDP $\Omega_{\mu_I}(M, \Delta)$ lies at the point ($M = 0$, $\Delta = M_0$). This means that in the model (1) the isospin symmetry is always broken down and a charged pion condensate which is equal to the quark mass M_0 in the vacuum, is created if $\mu_I > 0$.

Since in the vacuum case ($\mu = 0$, $\mu_I = 0$) chiral symmetry is spontaneously broken down in the model (1), there must exist three massless Nambu-Goldstone bosons which are pions, i.e. $m_\pi = 0$. So, we have proved that in the framework of the model (1) the pion condensation phase is realized at $\mu_I > m_\pi$, where m_π is the pion mass in the vacuum. Just the same phase structure is predicted by QCD at $\mu = 0$, $\mu_I \neq 0$ [5]. In contrast, in the framework of (3 + 1)-dimensional NJL-type models the pion condensation is not allowed for sufficiently high values of the isospin chemical potential [6–8]. This fact supports the statement made in the introduction that the NJL approach is only valid at rather small energies (chemical potentials). Moreover, we have once more demonstrated that in the leading order of the large-$N_c$expansion the two-dimensional GN models are a quite good theoretical laboratory for qualitative QCD investigations. So we are in a position to believe that the results obtained in the next sections are also inherent to QCD.

III. THE CASE OF $R^1 \times S^1$ SPACETIME AND $\mu_I \neq 0$

In the present section we continue the investigation of the charged pion condensation, this time under the influence of the finite volume occupied by the system. This is obviously a reasonable task, since all physical effects take place in restricted space regions. The consideration of the problem is significantly simplified in the framework of the two-dimensional model (1) at $\mu_I \neq 0$, which is again justified by its similarity to QCD. So we put a system with Lagrangian (1) into a restricted space region of the form $0 \leq x \leq L$ (here x is the space coordinate). It is well known that in this case the consideration is equivalent to the investigation of the model in a spacetime with nontrivial topology $R^1 \times S^1$ and with quantum fields, satisfying some boundary conditions of the form

$$q(t, x + L) = e^{i\pi\alpha} q(t, x), \tag{21}$$

where $0 \leq \alpha < 2$, L is the length of the circumference S^1, and the variable x means the path along it. Below, we shall use only two values of the parameter α: $\alpha = 0$ for periodic boundary conditions and $\alpha = 1$ for the antiperiodic one.

As a consequence, to obtain the thermodynamic potential $\Omega_{L\mu_I}(M, \Delta)$ of the initial system placed in the restricted domain $0 \leq x \leq L$ and at $\mu_I \neq 0$, one must simply replace the integration in (17) by an infinite series, using the rule:

$$\int_{-\infty}^{\infty} \frac{dp_1}{2\pi} f(p_1) \to \frac{1}{L} \sum_{n=-\infty}^{\infty} f(p_{1n}), \qquad p_{1n} = \frac{\pi}{L}(2n + \alpha), \qquad n = 0, \pm 1, \pm 2, \ldots \tag{22}$$

Moreover, instead of $V_0(\rho)$ it is necessary to use the effective potential $V_L(\rho)$ of the model in the vacuum (see the appendix). As a result, the TDP (17) will be replaced by the corresponding expression for the spacetime of the form $R^1 \times S^1$, i.e.

$$\Omega_{L\mu_I}(M, \Delta) = V_L(\rho) - \frac{1}{L} \sum_{n=-\infty}^{\infty} \Bigg\{ \sqrt{\Bigg(\sqrt{M^2 + \frac{\pi^2}{L^2}(2n + \alpha)^2} + \frac{\mu_I}{2}\Bigg)^2 + \Delta^2} + \sqrt{\Bigg(\sqrt{M^2 + \frac{\pi^2}{L^2}(2n + \alpha)^2} - \frac{\mu_I}{2}\Bigg)^2 + \Delta^2} - 2\sqrt{\rho^2 + \frac{\pi^2}{L^2}(2n + \alpha)^2} \Bigg\}, \tag{23}$$

where $\rho = \sqrt{M^2 + \Delta^2}$, and the function $V_L(\rho)$ is defined in (A10). In what follows, it will be convenient to use the dimensionless quantities

$$\lambda = \frac{\pi}{LM_0}, \qquad \nu = \frac{\mu_I}{2M_0}, \qquad m = \frac{M}{M_0}, \qquad \delta = \frac{\Delta}{M_0}, \qquad \mathfrak{O}_{\lambda\nu}(m, \delta) = \frac{\pi}{M_0^2} \Omega_{L\mu_I}(M, \Delta), \tag{24}$$

where M_0 is the dynamical quark mass in the vacuum. Moreover, since the phase structure of the model in the two particular cases $L = \infty$, $\mu_I \neq 0$ and $L \neq \infty$, $\mu_I = 0$ was already considered in Sec. II B and in the appendix, we will now investigate the phase structure only at $\lambda > 0$, $\nu > 0$.

A. The case of periodic boundary conditions

In this case $\alpha = 0$, and in terms of the dimensionless quantities (24) the TDP (23) can be rewritten in the following explicit form:

$$\begin{aligned} \mathfrak{O}_{\lambda\nu}(m, \delta) = {} & (m^2 + \delta^2)[\ln(4\lambda) - \gamma] - \lambda\sqrt{(m + \nu)^2 + \delta^2} - \lambda\sqrt{(m - \nu)^2 + \delta^2} \\ & - 2\lambda \sum_{n=1}^{\infty} \Bigg\{ \sqrt{\left(\sqrt{m^2 + (2n\lambda)^2} + \nu\right)^2 + \delta^2} + \sqrt{\left(\sqrt{m^2 + (2n\lambda)^2} - \nu\right)^2 + \delta^2} - 4n\lambda - \frac{m^2 + \delta^2}{2n\lambda} \Bigg\}, \end{aligned} \tag{25}$$

where $\gamma = 0.577\ldots$ is the Euler's constant [29]. We consider the TDP (25) as a function of two variables, m, δ. Moreover, ν, λ are free parameters there. Since the information about the phase structure of the model in the case of the periodic boundary conditions is contained in the global minimum point of the function (25) vs m, δ, it is first of all necessary to study the gap equations and then to investigate the behavior of the global minimum point vs parameters ν, λ. In particular, it is possible to show that for each fixed point of the plane (λ, ν) (with $\nu > 0$ and $\lambda \geq 0$) the global minimum point of the TDP (25) might be located at two different points only, (i) ($m = 0$, $\delta = 0$) and (ii) ($m = 0$, $\delta \neq 0$), where the nonzero gap δ is the solution of the equation $\partial\mathfrak{O}_{\lambda\nu}(0, \delta)/\partial(\delta^2) = 0$. The point (i) corresponds to the $U_{I_3}(1) \times U_{AI_3}(1)$ symmetric phase of the model (without charged pion condensation). On the other hand, if the global minimum of the function (25) is situated at the point (ii), then in the ground state of the model the isospin symmetry $U_{I_3}(1)$ is spontaneously broken down, and the pion condensation takes place. Let us denote by l_c the critical curve which separates the region of the (λ, ν) plane with symmetric phase from the points (λ, ν), corresponding to the pion condensed phase of the model. Since in each point of the curve l_c there is a phase transition of the second order from the symmetric phase to the pion condensed one and vice versa, the gap δ must vanish on this curve. So the critical curve l_c is defined by the following equation:

$$l_c\colon \left.\frac{\partial\mathfrak{O}_{\lambda\nu}(m, \delta)}{\partial(\delta^2)}\right|_{m,\delta=0} \equiv \ln(4\lambda) - \gamma - \frac{\lambda}{\nu} - \sum_{n=1}^{\infty} \Bigg\{ \frac{\lambda}{2n\lambda + \nu} + \frac{\lambda}{|2n\lambda - \nu|} - \frac{1}{n} \Bigg\} = 0. \tag{26}$$

To represent the curve l_c in the plane (λ, ν), it is convenient to divide this plane into an infinite set of regions ω_k:

$$(\lambda, \nu) = \bigcup_{k=1}^{\infty} \omega_k; \qquad \omega_k = \{(\lambda, \nu)\colon 2\lambda(k-1) \le \nu \le 2\lambda k\}. \tag{27}$$

In accordance with the division (27), the critical curve l_c can also be presented as a set of pieces, $l_c = \bigcup_{k=1}^{\infty} l_{ck}$. Obviously, each piece l_{ck} of the whole critical curve l_c lies inside the corresponding kth region ω_k and obeys the following equation ($k > 1$):

$$l_{ck}\colon\ \ln(4\lambda) - \gamma - \frac{\lambda}{\nu} - \sum_{n=1}^{k-1}\left\{\frac{\lambda}{2n\lambda+\nu} + \frac{\lambda}{\nu-2n\lambda} - \frac{1}{n}\right\} - \sum_{n=k}^{\infty}\left\{\frac{\lambda}{2n\lambda+\nu} + \frac{\lambda}{2n\lambda-\nu} - \frac{1}{n}\right\} = 0. \tag{28}$$

For $k = 1$ the part l_{c1} obeys Eq. (26) with omitted absolute value symbols. Performing the summations in (26) or (28), one can find for each piece l_{ck} of the critical curve l_c the following equation ($k \ge 1$):

$$l_{ck}\colon\ 2\ln(4\lambda) + 2\psi\left(k - \frac{\nu}{2\lambda}\right) - \psi\left(1 - \frac{\nu}{2\lambda}\right) + \psi\left(\frac{\nu}{2\lambda}\right) = 0, \tag{29}$$

which is valid only at $2\lambda(k-1) \le \nu \le 2\lambda k$. Here $\psi(x)$ is the logarithmic derivative of the Euler's $\Gamma(x)$ function [29]. Before drawing the critical curve l_c, we would like to point out its peculiarity. Using the well-known property of the $\psi(x)$ function, $\pi\cot(\pi x) = \psi(1-x) - \psi(x)$, as well as the periodicity of $\cot(\pi x)$, Eq. (29) can be reduced to the following one:

$$l_{ck}\colon\ 2\ln(4\lambda) = -2\psi(z) - \pi\cot(\pi z) \equiv F(z), \tag{30}$$

where $z = k - \nu/(2\lambda)$ and $0 \le z \le 1$. Since the absolute minimum of the function $F(z)$ from (30) corresponds to the point $z = 1/2$, each branch l_{ck} of the critical curve lies to the right of the vertical line $\lambda = \lambda_c$ [in the plane (λ, ν)], where $2\ln(4\lambda_c) = F(1/2)$, i.e. $\lambda_c = e^{\gamma} \approx 1.78$. All the branches of the critical curve l_c as well as the phase portrait of the initial model in terms of (λ, ν) are presented in Fig. 1 (left picture). Clearly, there is a strip $0 \le \lambda < \lambda_c$ which lies, as a whole, inside the region, corresponding to the pion condensed phase.

In the right picture of Fig. 1 the behavior of the pion condensation gap δ vs λ is depicted at $\nu = 7.5$. It is easily seen that this quantity oscillates as a function of λ. However, the amplitude of this oscillation is a rapidly decreasing function of λ when $\lambda \to 0$. Similar oscillations of different physical quantities such as gaps, critical curves, particle densities, etc. vs λ were also observed in some NJL-type models with one compactified space coordinate, but in a qualitatively alternative case with nonzero baryonic chemical potential [30]. Moreover, oscillating phenomena as functions of curvature are inherent to NJL models in the Einstein universe, i.e. in the curved space-time of the form $R^1 \times S^3$ [31]. In Fig. 2 the behavior of the gap δ vs ν is depicted at $\lambda = 1$ (left picture) and $\lambda = 1.7$ (right picture). Concerning this type of oscillations of the gap δ, it is necessary to note first of all that its period is equal to 2λ. Moreover, it is clear from Fig. 2, and this fact is supported by numerical calculations, that the amplitude of the oscillations of the quantity δ vs ν is a very slowly decreasing function of ν. Finally, it is evident that the smaller λ, the smaller the amplitude of this ν oscillations of the gap δ.

B. The case of antiperiodic boundary conditions

In this case $\alpha = 1$, so in (23) instead of V_L one should use the effective potential (A12). Then in terms of the

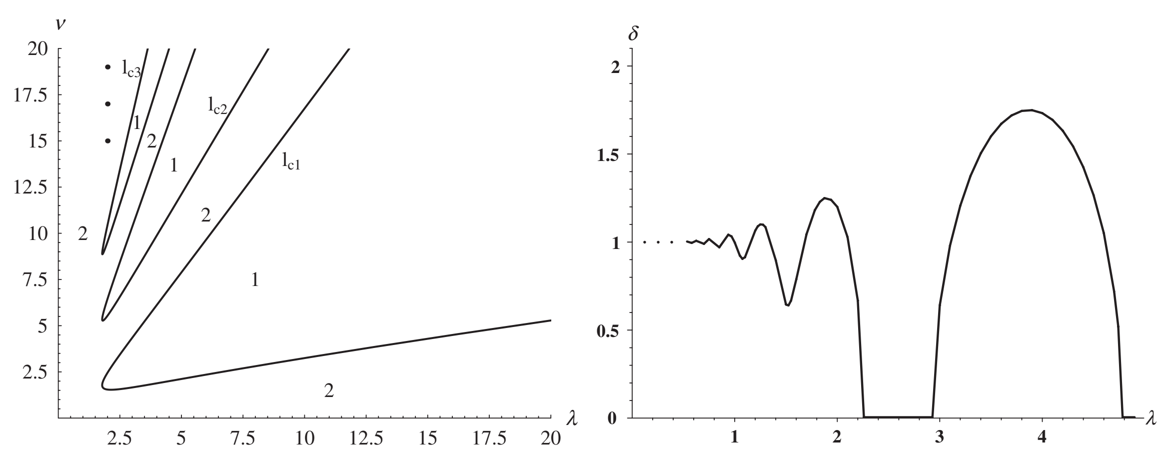

FIG. 1. *The periodic case*: In the left picture the phase portrait of the model is represented in terms of (λ, ν), where number 1 denotes the symmetric phase and number 2 denotes the pion condensed phase. In the right picture the behavior of the gap δ vs λ is depicted at $\nu = 7.5$.

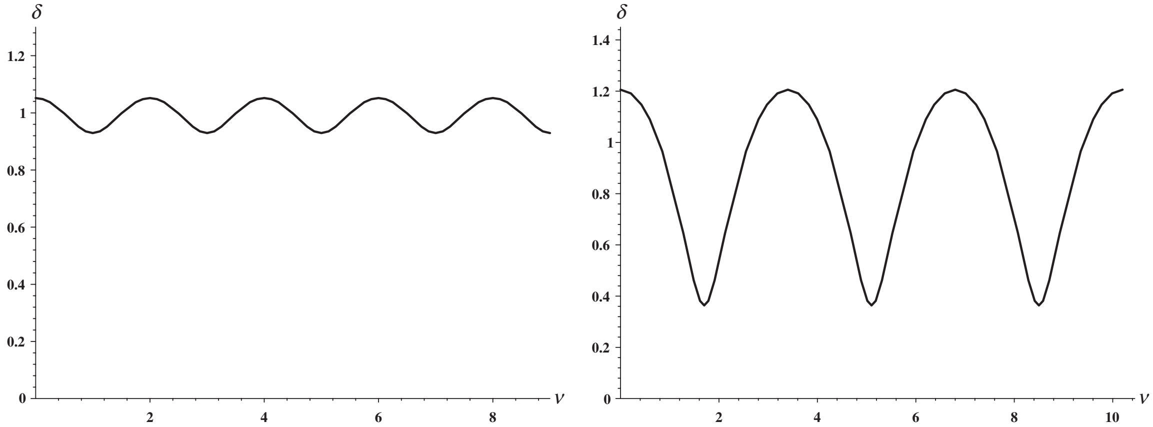

FIG. 2. *The periodic case*: The behavior of the gap δ vs ν at $\lambda = 1$ (left picture) and $\lambda = 1.7$ (right picture).

quantities (24) we have

$$\mathfrak{O}_{\lambda\nu}(m,\delta) = (m^2+\delta^2)[\ln(\lambda)-\gamma] - 2\lambda\sum_{n=0}^{\infty}\bigg\{\sqrt{(\sqrt{m^2+(2n+1)^2\lambda^2}+\nu)^2+\delta^2} + \sqrt{(\sqrt{m^2+(2n+1)^2\lambda^2}-\nu)^2+\delta^2} - 2(2n+1)\lambda - \frac{m^2+\delta^2}{(2n+1)\lambda}\bigg\}. \tag{31}$$

The critical curve l_c which divides the parameter plane (λ, ν) into a region with symmetric phase and the region, corresponding to a pion condensed phase, is now defined by the following equation:

$$l_c\colon \frac{\partial\mathfrak{O}_{\lambda\nu}(m,\delta)}{\partial(\delta^2)}\bigg|_{m,\delta=0} \equiv \ln(\lambda)-\gamma-\sum_{n=0}^{\infty}\bigg\{\frac{\lambda}{(2n+1)\lambda+\nu} + \frac{\lambda}{|(2n+1)\lambda-\nu|} - \frac{2}{2n+1}\bigg\} = 0. \tag{32}$$

As in the case with periodic boundary conditions, for solving Eq. (32) it is convenient to represent the parameter (λ, ν)-plane as the union of ω_k regions, $(\lambda,\nu) = \bigcup_{k=0}^{\infty}\omega_k$, where

$$\omega_0 = \{(\lambda,\nu)\colon 0\le\nu\le\lambda\},$$
$$\omega_k = \{(\lambda,\nu)\colon (2k-1)\lambda\le\nu\le(2k+1)\lambda\} \quad \text{for } k\ge 1. \tag{33}$$

Accordingly, in this case the critical curve l_c is composed of different pieces, i.e. $l_c = \bigcup_{k=0}^{\infty} l_{ck}$, where l_{ck} is the part of l_c, arranged in the corresponding region ω_k. Obviously, the equation for l_{c0} is just Eq. (32) with omitted absolute value symbols. However, the equations for l_{ck} $(k\ge 1)$ look like

$$l_{ck}\colon \ln(\lambda)-\gamma-\sum_{n=1}^{k-1}\bigg\{\frac{\lambda}{(2n+1)\lambda+\nu}+\frac{\lambda}{\nu-(2n+1)\lambda} - \frac{2}{2n+1}\bigg\} - \sum_{n=k}^{\infty}\bigg\{\frac{\lambda}{(2n+1)\lambda+\nu} + \frac{\lambda}{(2n+1)\lambda-\nu}-\frac{2}{(2n+1)}\bigg\} = 0. \tag{34}$$

Summing in (32) or (34) with the help of a program of analytical calculations, it is possible to obtain the more concise form of the equations for different pieces l_{ck} $(k\ge 0)$ of the critical curve:

$$l_{ck}\colon 2\ln(4\lambda)+2\psi\Big(k+\frac{1}{2}-\frac{\nu}{2\lambda}\Big)-\psi\Big(\frac{1}{2}-\frac{\nu}{2\lambda}\Big) + \psi\Big(\frac{1}{2}+\frac{\nu}{2\lambda}\Big) = 0. \tag{35}$$

As in Sec. III A, Eq. (35) can be transformed to a formally ω_k-independent expression

$$l_{ck}\colon 2\ln(4\lambda) = -2\psi(z)-\pi\cot(\pi z), \tag{36}$$

where $z = k+\frac{1}{2}-\nu/(2\lambda)$ and $0\le z\le\frac{1}{2}$ for $k=0$, whereas $0\le z\le 1$ for $k\ge 1$. Note that Eq. (36) coincides with (30) except for the different ν, λ dependence of the variable z. In Fig. 3 (left picture) the first several branches l_{ck} of the whole critical curve l_c, which divides the (λ,ν) plane into a region with pion condensed phase (the number 2 in the figure) and a region corresponding to a symmetric phase (the number 1 in the figure), are represented. Note that the strip $0\le\lambda\le\lambda_c$ of the plane belongs to the region 2 with the pion condensation phase.

In the right picture of Fig. 3 as well as in Fig. 4 the oscillating behavior of the pion condensation gap δ vs λ and, correspondingly, vs ν is depicted. The properties of these oscillations are the same as in the periodic case. Namely, at fixed ν the gap δ is a quickly damping oscillating function of λ when $\lambda\to 0$, whereas at fixed λ the gap δ oscillates at $\nu\to\infty$ with a very weak damping.

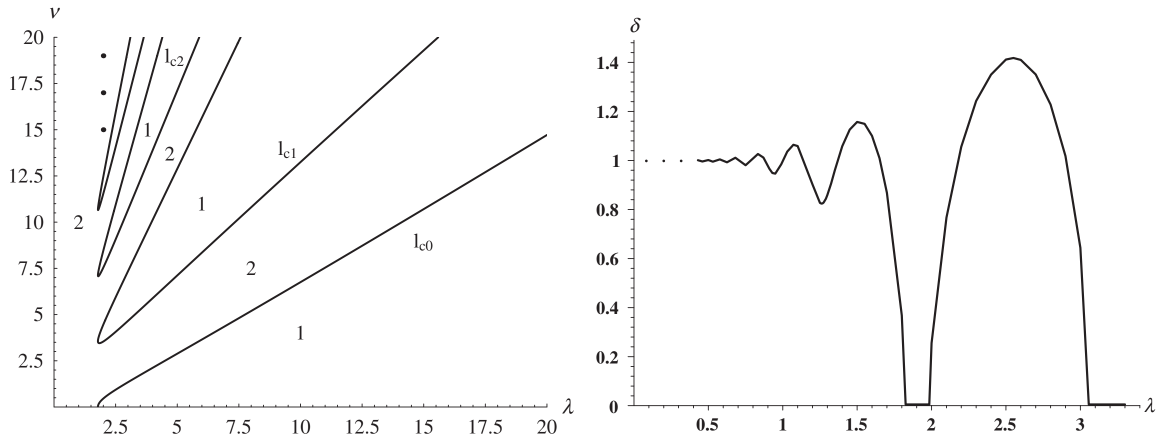

FIG. 3. *The antiperiodic case*: The phase portrait (left picture) and the gap δ at $\nu = 7.5$ (right picture). The notations are the same as in Fig. 1.

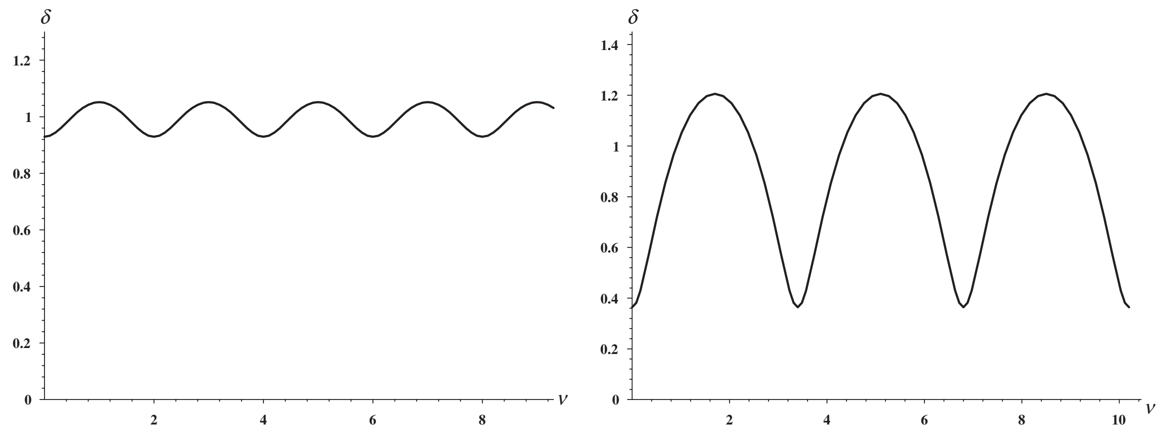

FIG. 4. *The antiperiodic case*: The behavior of the gap δ vs ν at $\lambda = 1$ (left picture) and $\lambda = 1.7$ (right picture).

IV. SUMMARY AND DISCUSSION

In the present paper we have studied the phase structure of a two-dimensional GN model at nonzero isospin chemical potential μ_I and in the spacetime $R^1 \times S^1$ with nontrivial topology, when the space coordinate is compactified into a circumference of a finite length L. The consideration is performed in the leading order of the large-N_c expansion technique.[2]

It turns out that in the case with $L = \infty$ the pion condensed phase is realized in the model at arbitrary nonzero values of μ_I. In this phase the corresponding order parameter, the pion condensate Δ, does not depend on the isospin chemical potential μ_I and is equal to M_0, i.e. to the dynamical quark mass in the vacuum. The same phase structure at $\mu_I \neq 0$ occurs in the chirally symmetric QCD, where pions are massless particles, so one more common property is found which is shared both by the GN model and QCD. As a result, the assurance that the finite size ($L \neq \infty$) effects of the GN model are also inherent to compactified QCD at $\mu_I \neq 0$ is raised.

If L is finite, then the phase portraits of the model in terms of $\lambda \sim 1/L$ and $\nu \sim \mu_I$ are found for the case of periodic (see Fig. 1) and antiperiodic (see Fig. 3) boundary conditions. Among the most interesting properties of these phase diagrams is the fact that the strip $0 \leq \lambda < \lambda_c \approx 1.78$ lies as a whole inside the pion condensed phase. We have shown also that the pion condensed gap δ is an oscillating function vs both λ (at fixed ν) and ν (at fixed λ). The same is true for other thermodynamic quantities of the model such as pressure, particle densities, etc., and is inherent also to the $(3 + 1)$-dimensional NJL models with curved spacetimes [31] or spacetimes with nontrivial topology [30].

[2]It should be noted that the problem of the IR-behavior of the correlation function of quantum fluctuations in two-dimensional quantum field theory (QFT) models was extensively discussed in literature with relation to the Coleman-Mermin-Wagner theorem. One may mention the papers on the 2-dimensional GN model [11,15–17], where this problem has been investigated and it was demonstrated that for the limit of infinite N_c this theorem is not valid, and hence spontaneous symmetry breaking may take place.

One more interesting issue should also be mentioned. Although in the present paper the spatially uniform pion condensation was assumed for simplicity, however, for sufficiently high values of μ_I, the pion superfluidity with inhomogeneous condensate might be realized in isotopically asymmetric and spatially infinite quark matter systems [32]. A detailed investigation of this possibility in the case of finite space volume is outside the scope of this paper and should be left for further studies in the framework of different QCD-like models including the GN model.

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APPENDIX: EFFECTIVE POTENTIAL IN THE VACUUM ($\mu_I = 0$)

Note that in the vacuum and in the $R^1 \times R^1$ spacetime, the expression V_0 for the effective potential of the initial model (1) can be found starting from the TDP (11) at $\mu = \mu_I = 0$, where without loss of generality it is possible to put $\pi_a = 0$ ($a = 1, 2, 3$):

$$V_0(\sigma) = \frac{\sigma^2}{4G} + i\int \frac{d^2p}{(2\pi)^2} \ln[(p_0^2 - p_1^2 - \sigma^2)^2] = \frac{\sigma^2}{4G} - 2\int_{-\infty}^{\infty} \frac{dp_1}{2\pi}\sqrt{\sigma^2 + p_1^2}. \tag{A1}$$

Here the second equality is obtained with the help of formula (13). Since in (A1) the last integral is an UV divergent one, we regularize it by cutting off the integration region, i.e. supposing that $|p_1| < \Lambda$. The effective potential $V_0(\sigma)$, as a whole, must be a finite quantity at $\Lambda \to \infty$. So the UV divergence of the integral term in (A1) must be compensated by the term with the bare coupling constant G, which, of course, has to be a Λ-dependent quantity. To find an appropriate expression for G, let us recall that in the vacuum the chiral symmetry is necessarily broken down in the framework of the model (1), and quarks acquire a nonzero dynamical mass M_0 which is a nontrivial solution of the gap equation $V_0'(\sigma) = 0$. Taking into account this circumstance, one can immediately obtain from the gap equation the expression (19) for the bare coupling G. Substituting it in (A1), it is possible to get for $\Lambda \gg M_0$ the effective potential (20) of the initial model in the vacuum and in the $R^1 \times R^1$ spacetime. (More details about the above renormalization procedure for $V_0(\sigma)$ are presented, e.g., in [21,23]).

Now let us find the effective potential $V_L(\sigma)$ of the model (1), when the spacetime has a nontrivial topology of the form $R^1 \times S^1$ and quark fields obey the most general boundary conditions (21). In this case one can start from Eq. (A1), in which it is necessary to perform the Euclidian rotation ($p_0 \to ip_0$) and then use the transformations according to the rule (22). As a result, we have

$$V_L(\sigma) = \frac{\sigma^2}{4G} - \frac{2}{L}\sum_{n=-\infty}^{\infty}\int \frac{dp_0}{2\pi} \ln\left[p_0^2 + \sigma^2 + \frac{\pi^2}{L^2}(2n+\alpha)^2\right]. \tag{A2}$$

Let us next use in (A2) the formula $\ln a = -\int_0^\infty \frac{ds}{s} e^{-as}$, which is valid up to an infinite constant independent of the parameter a, as well as the Poisson sum formula [29]

$$\sum_{n=-\infty}^{\infty} e^{-s(\pi^2/L^2)(2n+\alpha)^2} = \frac{L}{2\pi}\sqrt{\frac{\pi}{s}}\sum_{n=-\infty}^{\infty} e^{-(n^2L^2/4s)} e^{in\pi\alpha} = \frac{L}{2\pi}\sqrt{\frac{\pi}{s}}\left\{1 + 2\sum_{n=1}^{\infty} e^{-(n^2L^2/4s)} \times \cos(n\pi\alpha)\right\}. \tag{A3}$$

After integration over p_0, one can easily find

$$V_L(\sigma) = V_0(\sigma) + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^\infty \frac{ds}{s^2} e^{-\sigma^2 s - (n^2L^2/4s)} \cos(n\pi\alpha), \tag{A4}$$

where

$$V_0(\sigma) = \frac{\sigma^2}{4G} + \frac{1}{2\pi}\int_0^\infty \frac{ds}{s^2} e^{-\sigma^2 s} \tag{A5}$$

is another, equivalent, expression for the effective potential (A1). [It is clear from general considerations that at $L \to \infty$ the effective potential $V_L(\sigma)$ must coincide with the effective potential in the $R^1 \times R^1$ spacetime, i.e. with $V_0(\sigma)$ given in (A1). Looking at the formula (A4) (or (A7) below) at $L \to \infty$, one can easily obtain in the right-hand side the expression appearing in the right-hand side of formula (A5). So, due to the above reason, it is the function $V_0(\sigma)$.] Moreover, in the following we will use for $V_0(\sigma)$ its renormalized expression (20). Let us integrate in (A4) over s using the well-known relation [33]

$$\int_0^\infty ds s^{n-1} e^{-(A/s) - Bs} = 2\left(\frac{A}{B}\right)^{n/2} K_n(2\sqrt{AB}), \tag{A6}$$

where $K_n(z) = K_{-n}(z)$ is the Macdonald function [29]. Then,

$$V_L(\sigma) = V_0(\sigma) + \frac{4\sigma}{\pi}\sum_{n=1}^{\infty}\frac{\cos(n\pi\alpha)}{nL}K_1(nL\sigma). \tag{A7}$$

Because of the relation $z\frac{d}{dz}K_1(z) = -K_1(z) - zK_0(z)$, it follows from (A7) that

$$\frac{\partial}{\partial\sigma}V_L(\sigma) = \frac{\partial}{\partial\sigma}V_0(\sigma) - \frac{4\sigma}{\pi}\sum_{n=1}^{\infty}\cos(n\pi\alpha)K_0(nL\sigma). \tag{A8}$$

The series in (A8) can be modified appropriately [33], so [here we use the effective potential $V_0(\sigma)$ in its renormalized form (20)]

$$\begin{aligned}\frac{\partial}{\partial\sigma}V_L(\sigma) = & -\frac{2\sigma}{\pi}\ln\left(\frac{M_0L}{4\pi}\right) - \frac{2\sigma\gamma}{\pi} - \frac{2\sigma}{\sqrt{\sigma^2L^2+\pi^2\alpha^2}} \\ & - 2\sigma\sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{\pi^2(2n+\alpha)^2+L^2\sigma^2}} - \frac{1}{2n\pi}\right] \\ & - 2\sigma\sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{\pi^2(2n-\alpha)^2+L^2\sigma^2}} - \frac{1}{2n\pi}\right],\end{aligned} \tag{A9}$$

where $\gamma = 0.577\ldots$ is the Euler constant. Integrating both sides of (A9) over σ, we obtain the final expression for the effective potential $V_L(\sigma)$ of the initial GN model in the vacuum, when the space coordinate is compactified

$$\begin{aligned}V_L(\sigma) - V_L(0) = & -\frac{\sigma^2}{\pi}\ln\left(\frac{M_0L}{4\pi}\right) - \frac{\sigma^2\gamma}{\pi} \\ & - \frac{2}{L^2}\sqrt{\sigma^2L^2+\pi^2\alpha^2} + \frac{2\pi\alpha}{L^2} \\ & - \frac{2}{L^2}\sum_{n=1}^{\infty}\Bigg[\sqrt{\pi^2(2n+\alpha)^2+L^2\sigma^2} \\ & + \sqrt{\pi^2(2n-\alpha)^2+L^2\sigma^2} \\ & - 4n\pi - \frac{\sigma^2L^2}{2n\pi}\Bigg].\end{aligned} \tag{A10}$$

In spite of the fact that (A10) is valid for arbitrary $0 \leq \alpha < 2$, we find it more convenient to have another (equivalent) expression for the function $V_L(\sigma)$ at $\alpha = 1$. In this case one can start again from the relation (A9), in which in the second series it is necessary to shift the summation index, i.e. $n \to n+1$. Then, manipulating with convergent infinite sums, we obtain

$$\begin{aligned}\frac{\partial}{\partial\sigma}V_L^{\alpha=1}(\sigma) = & -\frac{2\sigma}{\pi}\ln\left(\frac{M_0L}{\pi}\right) - \frac{2\sigma\gamma}{\pi} \\ & - 4\sigma\sum_{n=0}^{\infty}\Bigg[\frac{1}{\sqrt{\pi^2(2n+1)^2+L^2\sigma^2}} \\ & - \frac{1}{(2n+1)\pi}\Bigg].\end{aligned} \tag{A11}$$

Now, in order to get the effective potential, one should integrate both sides of this relation over σ:

$$\begin{aligned}V_L^{\alpha=1}(\sigma) - V_L^{\alpha=1}(0) = & \frac{\sigma^2}{\pi}\ln\left(\frac{\pi}{M_0L}\right) - \frac{\sigma^2\gamma}{\pi} - \frac{4}{L^2} \\ & \times\sum_{n=0}^{\infty}\Bigg[\sqrt{\pi^2(2n+1)^2+L^2\sigma^2} \\ & - (2n+1)\pi - \frac{\sigma^2L^2}{2(2n+1)\pi}\Bigg].\end{aligned} \tag{A12}$$

Finally, a few words about the phase structure of the model at finite L. It is clear that if the antiperiodic boundary conditions are imposed on the quark fields, i.e. $\alpha = 1$ in (21), the parameter L plays the role of the inverse temperature in the ordinary GN model, but only in the vacuum (with zero chemical potentials). In the last case the critical properties of the GN model are well understood (see, e.g., in [34]), so, by analogy, we can conclude that at $L < L_c = \frac{\pi}{M_0}e^{-\gamma}$ the global minimum of the effective potential (A12) lies at the point $\sigma = 0$. In this case the chiral symmetry of the initial GN (1) model is not broken. In contrast, at $L > L_c$ the effective potential (A12) has a nontrivial global minimum point. As a result, chiral symmetry is spontaneously broken down at sufficiently high L.

The situation is, however, quite different at periodic boundary conditions, i.e. if $\alpha = 0$ in (21). In this case the global minimum point of the corresponding effective potential (A10) lies outside the point $\sigma = 0$ for all $L \neq 0$. Indeed, it is clear from (A9) that the derivative of this function is negative at sufficiently small values of σ, so there is always a local maximum of the effective potential at the point $\sigma = 0$, and the chirally broken phase is realized in the model for arbitrary $L \neq 0$.

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