Finite-size effects in pion condensation phenomena of dense baryonic matter in the NJL_2 model

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The properties of two-flavored massless Nambu-Jona-Lasinio model in (1+1)-dimensional $R^1 \times S^1$ spacetime with compactified space coordinate are investigated in the presence of isospin and quark number chemical potentials μ_I, μ. The consideration is performed in the large N_c limit, where N_c is the number of colored quarks. It is shown that at $L = \infty$ (L is the length of the circumference S^1) the pion condensation (PC) phase with *zero quark number density* is realized at arbitrary nonzero μ_I and for rather small values of μ. However, at arbitrary finite values of L the phase portrait of the model contains the PC phase with *nonzero quark number density* (in the case of periodic boundary conditions for quark fields). Hence, finite sizes of the system can serve as a factor promoting the appearance of the PC phase in quark matter with nonzero baryon densities. In contrast, the phase with chiral symmetry breaking may exist only at rather large values of L.

I. INTRODUCTION

Recently, great attention has been paid to the investigation of the QCD phase diagram in terms of baryonic as well as isotopic (isospin) chemical potentials. The reason is that dense baryonic matter which can appear in heavy-ion collision experiments has an evident isospin asymmetry. Moreover, the dense hadronic/quark matter inside compact stars is in general also expected to be isotopically asymmetric. To describe the above mentioned realistic situations, i.e. when the baryonic density is comparatively low, usually different nonperturbative methods or effective theories such as chiral effective Lagrangians and especially Nambu – Jona-Lasinio (NJL) type models [1] are employed. In this way the QCD phase diagram including chiral symmetry restoration [2–6], color superconductivity (CSC) [7–9], and charged pion condensation (PC) phenomena [10–13] were investigated under heavy-ion experimental and/or compact star conditions, i.e. in the presence of such external conditions as temperature, chemical potentials and possible external (chromo)magnetic fields (see the above references).

Moreover, it was also realized in the framework of NJL-type models that different finite size factors, i.e. curvature or nontrivial spacetime topology as well as a finite spatial volume of a system, can significantly change the properties of both the chiral- and color superconductivity phase transitions. This conclusion is the result of chiral symmetry breaking investigations in weakly curved spaces [14, 15] and in spaces with nontrivial topology, where one or more space coordinates are compactified [16]. In addition, the properties of finite size normal quark matter droplets in the language of the MIT-bag model were considered, e.g., in the review [17]. It was also noted that the position of the chiral critical end point of the QCD phase diagram, which could be investigated in heavy ion collision experiments, depends essentially on the size of a finite system [18]. Next, the effect of spaces with constant curvature or nontrivial topology on CSC was studied in [19, 20]. In particular, there it was shown that in a restricted region the CSC phenomenon might occur much easier than in an infinite one [20]. Moreover, the stability of finite size quark matter droplets in the color-flavor locked phase was considered in the framework of a bag model using the so-called multiple expansion method [21]. However, up to now we have only scarce information about the formation of the pion condensation phase in dense baryonic matter under the influence of finite-size effects (see below). This fact is partially explained by the complexity of the problem arising from the necessity to introduce an additional isotopic chemical potential for the adequate description of quark matter with isospin asymmetry and, in particular, of the PC phenomenon.

Obviously, the (3+1)-dimensional NJL models depend on the cutoff parameter which is typically chosen of the order of one GEV, so that the results of their usage are valid only at *comparatively low energies, temperatures and densities (chemical potentials)*. Besides, there exists also a class of renormalizable theories, the (1+1)-dimensional chiral Gross-Neveu (GN) type models [22], [1] that can be used as a laboratory for the qualitative simulation of specific properties of QCD at *arbitrary energies*. Renormalizability, asymptotic freedom, as well as the spontaneous breaking of chiral symmetry (in vacuum) are the most fundamental inherent features both for QCD and all GN type models. In addition, the $\mu - T$ phase diagram is qualitatively the same for the QCD and GN model [24–27] (here μ is the quark number chemical potential and T is the temperature). Note also that the GN type models are suitable for the description of physics in quasi one-dimensional condensed matter systems like polyacetylene [28]. It is currently well understood (see, e.g., the discussion in [25–27]) that the usual no-go theorem [29], which generally forbids the spontaneous breaking of any continuous symmetries in two-dimensional spacetime does not work in the limit $N_c \to \infty$, where N_c is the number of colored quarks. This follows from the fact that in the limit of large N_c the quantum fluctuations, which

[1] Below we shall use the notation "NJL_2 model" instead of "chiral GN model" for (1+1)-dimensional models with *a continuous chiral symmetry*, since the chiral structure of the Lagrangian is the same as that of the (3+1)-dimensional NJL model.

would otherwise destroy a long-range order corresponding to a spontaneous symmetry breaking, are suppressed by $1/N_c$ factors. Thus, the effects inherent for real dense quark matter, such as color superconductivity (spontaneous breaking of the continuous color symmetry) or charged pion condensation (spontaneous breaking of the continuous isospin symmetry) might be simulated in terms of a simpler (1+1)-dimensional GN-type model, though only in the leading order of the large N_c approximation (see, e.g., [26, 30] and [31, 32], respectively). Finally, one should recall that both the chiral phase transition [33] and color superconductivity [30] were investigated in the framework of GN models with account of the nontrivial $R^1 \times S^1$ spacetime topology.

In general, this paper is devoted to the consideration of the charged pion condensation phenomenon under the influence of finite-size effects. The problem was partially solved in [34], where PC was studied in the framework of the (3+1)-dimensional NJL model in the Einstein universe with a constant curvature and finite spatial volume, and in [31], where it was considered in the $SU_L(2) \times SU_R(2)$ symmetric GN model in spacetime with nontrivial topology, i.e. on the $R^1 \times S^1$ manifold with compactified space coordinate, and in the assumption that $N_c \to \infty$. For technical simplifications, in those papers the quark chemical potential μ was assumed to be zero, i.e. the considerations were performed at nonzero isospin μ_I chemical potential only (this situation corresponds to quark matter with zero baryon density). Obviously, the considered problem deserves further, more detailed investigations, this time when both chemical potentials are nonvanishing, $\mu \neq 0, \mu_I \neq 0$. This is physically motivated by the fact that quark matter, which might be created in heavy-ion collisions, has in general nonzero baryon- and isospin densities and must be investigated in the framework of a theory with nonzero μ and μ_I. Moreover, quark-matter lumps occupy a finite volume. Hence, in this paper we study, for illustration, the PC phenomenon in a $SU_L(2) \times SU_R(2)$ symmetric two-dimensional NJL$_2$ model with $\mu \neq 0, \mu_I \neq 0$, when the spatial coordinate is compactified and $N_c \to \infty$. In particular, we shall demonstrate that the finite size of the system promotes the appearance of the PC phenomenon in dense baryonic matter. We hope that the results of such a study of a renormalizable two-dimensional model may provide an additional stimulus for further investigations of the charged PC phenomenon in more realistic models.

The paper is organized as follows. In Section II we derive, in the leading order of the large N_c-expansion, the expression for the thermodynamic potential of the two-flavored massless NJL$_2$ model with quark number chemical potential μ and isospin chemical potential μ_I for zero temperature in $R^1 \times R^1$- and in $R^1 \times S^1$ spacetimes.Then, in Section III the phase structure of the model is investigated both at a finite value of L and at $L \to \infty$, where L is the radius of the S^1 circumference. Finally, Section IV presents a summary and some concluding remarks.

II. DESCRIPTION OF THE MODEL AND ITS THERMODYNAMIC POTENTIAL

A. Effective action

We consider a (1+1)-dimensional NJL$_2$ model to mimic the phase structure of real dense quark matter with two quark flavors (u and d quarks). Its Lagrangian has the form:

$$\mathcal{L}_{q,\bar{q}} = \bar{q}\left[\gamma^\nu \mathrm{i}\partial_\nu + \mu\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0\right]q + \frac{G}{N_c}\left[(\bar{q}q)^2 + (\bar{q}\mathrm{i}\gamma^5\vec{\tau}q)^2\right], \tag{1}$$

where each quark field $q(x) \equiv q_{i\alpha}(x)$ is a flavor doublet ($i = 1, 2$ or $i = u, d$) and color N_c-plet ($\alpha = 1, \ldots, N_c$). Moreover, it is a two-component Dirac spinor (in (1) the summation over flavor, color, and spinor indices is implied); $\tau_k (k = 1, 2, 3)$ are Pauli matrices. The quark number chemical potential μ in (1) is responsible for the nonzero baryonic density of quark matter, whereas the isospin chemical potential μ_I is taken into account in order to study properties of quark matter at nonzero isospin densities (in this case the densities of u and d quarks are different). The Dirac gamma matrices in two-dimensional spacetime have the following form:

$$\gamma^0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \gamma^1 = \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad \gamma^5 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Let us consider the symmetries of the Lagrangian. If $\mu_I = 0$, the Lagrangian (1) is not only $SU(N_c)$ symmetric, but also invariant under transformations of the chiral $SU_L(2) \times SU_R(2)$ group. However, if $\mu_I \neq 0$, the latter symmetry is reduced to $U_{I_3L}(1) \times U_{I_3R}(1)$, where $I_3 = \tau_3/2$ is the third component of the isospin operator (here and above the subscripts L, R mean that the corresponding groups only act on left- and right-handed spinors, respectively). Obviously, this symmetry can also be presented as $U_{I_3}(1) \times U_{AI_3}(1)$, where $U_{I_3}(1)$ is a vector isospin subgroup, and $U_{AI_3}(1)$ is an axial isospin subgroup. Quarks are transformed under these subgroups as $q \to \exp(i\alpha\tau_3)q$ and $q \to \exp(i\alpha\gamma^5\tau_3)q$, respectively.

The linearized version of Lagrangian (1), which contains composite bosonic fields $\sigma(x)$ and $\pi_a(x) (a = 1, 2, 3)$, has the following form:

$$\mathcal{L}_{\sigma,\pi} = \bar{q}\left[\gamma^\nu \mathrm{i}\partial_\nu + \mu\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0 - \sigma - \mathrm{i}\gamma^5\pi_a\tau_a\right]q - \frac{N_c}{4G}\left[\sigma\sigma + \pi_a\pi_a\right]. \tag{3}$$

¿From Lagrangian (3) one obtains the following constraint equations for the bosonic fields:

$$\sigma(x) = -2\frac{G}{N_c}(\overline{q}q); \quad \pi_a(x) = -2\frac{G}{N_c}(\overline{q}\mathrm{i}\gamma^5\tau_a q). \tag{4}$$

Obviously, Lagrangian (3) is equivalent to Lagrangian (1), when using the constraint equations (4). Furthermore, it is clear from (4) that the bosonic fields are transformed under the isospin $U_{I_3}(2)$ and axial isospin $U_{AI_3}(2)$ subgroups in the following manner:

$$\begin{aligned} U_{I_3}(1):& \quad \sigma\to\sigma; \quad \pi_3\to\pi_3; \quad \pi_1\to\cos(2\alpha)\pi_1+\sin(2\alpha)\pi_2; \quad \pi_2\to\cos(2\alpha)\pi_2-\sin(2\alpha)\pi_1,\\ U_{AI_3}(1):& \quad \pi_1\to\pi_1; \quad \pi_2\to\pi_2; \quad \sigma\to\cos(2\alpha)\sigma+\sin(2\alpha)\pi_3; \quad \pi_3\to\cos(2\alpha)\pi_3-\sin(2\alpha)\sigma. \end{aligned} \tag{5}$$

To avoid the no-go theorem, which forbids the spontaneous breaking of continuous symmetries in the considered case of one space dimension, we restrict the discussion only to the leading order of the large N_c expansion (i.e. to the case $N_c\to\infty$), where this theorem is not valid [25–27]. In particular, the effective action $S_{\rm eff}[\sigma,\pi_a]$ can be found in this approximation through the relation:

$$e^{\mathrm{i}S_{\rm eff}[\sigma,\pi_a]} = \int[d\overline{q}][dq]e^{\mathrm{i}\int d^2x\mathcal{L}_{\sigma,\pi}}. \tag{6}$$

Carrying out the Gaussian-type path integration in (6) over fermion fields, we obtain

$$e^{\mathrm{i}S_{\rm eff}[\sigma,\pi_a]} = e^{-\mathrm{i}\frac{N_c}{4G}\int d^2x(\sigma^2+\pi_a^2)}\det \mathrm{D}. \tag{7}$$

In (7) we have used the notation $\mathrm{D}\equiv D\times I_c$, where I_c is the unit operator in the N_c-dimensional color space and

$$D = \mathrm{i}\gamma^\nu\partial_\nu + \mu\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0 - \sigma - \mathrm{i}\gamma^5\pi_a\tau_a. \tag{8}$$

Then using the general formula $\det\mathrm{D} = \exp\mathrm{Tr}_{csfx}\ln\mathrm{D}$ and taking the logarithm of (7), one obtains the following expression for the effective action in the leading order of the $1/N_c$-expansion:

$$S_{\rm eff}(\sigma,\pi_a) = -N_c\int\frac{\sigma^2+\pi_a^2}{4G}d^2x - \mathrm{i}N_c\mathrm{Tr}_{sfx}\ln D, \tag{9}$$

where the Tr-operator stands for the trace in spinor (s), flavor (f), and two-dimensional coordinate (x) spaces, respectively (the trace in color (c) space, obviously, equals N_c).

B. Thermodynamic potential

Starting from (9), one can define the thermodynamic potential (TDP) of the model at zero temperature $T=0$ in the mean-field approximation, i.e. in the leading order of the $1/N_c$-expansion [31]:

$$\begin{aligned} \Omega_{\mu\mu_I}(\sigma,\pi_a) &\equiv -\frac{S_{\rm eff}(\sigma,\pi_a)}{N_c\int d^2x}\bigg|_{\sigma,\pi_a=\rm const} = \frac{\sigma^2+\pi_a^2}{4G} + \mathrm{i}\frac{\mathrm{Tr}_{sfx}\ln D}{\int d^2x}\\ &= \frac{\sigma^2+\pi_a^2}{4G} + \mathrm{i}\mathrm{Tr}_{sf}\int\frac{d^2p}{(2\pi)^2}\ln(\gamma p + \mu\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0 - \sigma - \mathrm{i}\gamma^5\pi_a\tau_a), \end{aligned} \tag{10}$$

where the mean fields σ and π_a are $x-$independent quantities, and in the round brackets of (10) just the momentum space representation $\overline{D} = \gamma p + \mu\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0 - \sigma - \mathrm{i}\gamma^5\pi_a\tau_a$ of the Dirac operator D appears. [2] Obviously, $\mathrm{Tr}_{sf}\ln\overline{D} = \sum_i\ln\epsilon_i$, where summation over all four eigenvalues ϵ_i of the 4×4 matrix $\overline{D}$ is implied and

$$\epsilon_{1,2,3,4} = -\sigma \pm \sqrt{(p_0+\mu)^2 - p_1^2 - \pi_a^2 + \left(\frac{\mu_I}{2}\right)^2 \pm \mu_I\sqrt{(p_0+\mu)^2 - \pi_1^2 - \pi_2^2}}. \tag{11}$$

Hence,

$$\begin{aligned} \Omega_{\mu\mu_I}(\sigma,\pi_a) =& \frac{\sigma^2+\pi_a^2}{4G} + i\int\frac{d^2p}{(2\pi)^2}\ln(\epsilon_1\epsilon_2\epsilon_3\epsilon_4)\\ =& \frac{\sigma^2+\pi_a^2}{4G} + i\int\frac{d^2p}{(2\pi)^2}\ln\{[(p_0+\mu)^2-\varepsilon_+^2]\times[(p_0+\mu)^2-\varepsilon_-^2]\}, \end{aligned} \tag{12}$$

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[2] There exists also the interesting case of x-dependent mean fields describing chiral density waves [35]. For the following investigation of finite-size effects we restrict us, however, for simplicity, to the case of constant fields.

where

$$\varepsilon_{\pm}=\sqrt{\left(\sqrt{p_1^2+\sigma^2+\pi_3^2}\pm\frac{\mu_I}{2}\right)^2+\pi_1^2+\pi_2^2}. \tag{13}$$

It is clear that the TDP $\Omega_{\mu\mu_I}(\sigma,\pi_a)$ is symmetric under the transformations $\mu\to-\mu$ and/or $\mu_I\to-\mu_I$. So it is sufficient to consider only the region $\mu\geq 0$, $\mu_I\geq 0$. Taking into account this constraint and integrating in (12), one obtains the following expression for the TDP of the system:

$$\Omega_{\mu\mu_I}(\sigma,\pi_a)=\frac{\sigma^2+\pi_a^2}{4G}-\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\{\varepsilon_++\varepsilon_-+(\mu-\varepsilon_+)\theta(\mu-\varepsilon_+)+(\mu-\varepsilon_-)\theta(\mu-\varepsilon_-)\}, \tag{14}$$

where $\theta(x)$ is the Heaviside step function. To simplify the task, let us note that both the quasiparticle energies (11) and hence the TDP (14) depend effectively only on two combinations $(\pi_1^2+\pi_2^2)$ and $(\pi_3^2+\sigma^2)$ of the bosonic fields, which are invariants with respect to the $U_{I_3}(1)\times U_{AI_3}(1)$ group, as is easily seen from (5). In this case, without loss of generality, one can put $\pi_2=\pi_3=0$, and study the thermodynamic potential (14) TDP as a function of only two variables, $M\equiv\sigma$ and $\Delta\equiv\pi_1$. Then the TDP has the following form:

$$\Omega_{\mu\mu_I}(M,\Delta)=\frac{M^2+\Delta^2}{4G}-\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\{E_\Delta^++E_\Delta^-+(\mu-E_\Delta^+)\theta(\mu-E_\Delta^+)+(\mu-E_\Delta^-)\theta(\mu-E_\Delta^-)\}, \tag{15}$$

where

$$E_\Delta^{\pm}=\sqrt{\left(\sqrt{p_1^2+M^2}\pm\frac{\mu_I}{2}\right)^2+\Delta^2}\equiv\sqrt{(E\pm\nu)^2+\Delta^2};\quad\left(E=\sqrt{p_1^2+M^2},\nu=\frac{\mu_I}{2}\right). \tag{16}$$

Since we are going to study the phase diagram of the initial NJL$_2$ model, the system of gap equations is needed:

$$\frac{\partial\Omega_{\mu\mu_I}(M,\Delta)}{\partial M}=0;\quad\frac{\partial\Omega_{\mu\mu_I}(M,\Delta)}{\partial\Delta}=0. \tag{17}$$

The coordinates M and Δ of the global minimum point (GMP) of the TDP (15) provide two order parameters (gaps), which are proportional to the ground state expectation values $<\bar qq>$ and $<\bar qi\gamma^5\tau_1q>$, respectively. In this case the gap M is just a dynamical quark mass. Obviously, the pair of gaps M and Δ is a solution of the system (17). So, the GMP of the form $(M=0,\Delta=0)$ corresponds to the phase, in which the initial $U_{I_3}(1)\times U_{AI_3}(1)$ symmetry remains intact. If $M\neq 0,\Delta=0$ in the GMP, then the $U_{I_3}(1)$ symmetric phase is realized in the model. Finally, the GMP of the form $(M=0,\Delta\neq 0)$ corresponds to the $U_{AI_3}(1)$ symmetric phase, where charged pions are condensed and isospin symmetry, $U_{I_3}(1)$, is broken spontaneously. (Note that due to the zero value of the bare quark mass in the Lagrangian (1), the GMP of the form $(M\neq 0,\Delta\neq 0)$ does not appear for the TDP (15).) On the basis of the gaps M and Δ the following quantities can be introduced,

$$p_{0u}=E_\Delta^--\mu,\quad p_{0d}=E_\Delta^+-\mu,\quad p_{0\overline{u}}=-(E_\Delta^++\mu)\quad p_{0\overline{d}}=-(E_\Delta^-+\mu), \tag{18}$$

which are the energies of u-, d-, $\bar u$-, $\bar d$-quasiparticles (quarks) or one-fermion excitations of the corresponding phase. The quantities p_{0u},p_{0d} from (18) are the energies necessary for creation of u and d quarks with momentum p_1, whereas $p_{0\overline{u}},p_{0\overline{d}}$ are the energies necessary for annihilation of $\bar u$ and $\bar d$ antiquarks.

It is clear that the TDP (15) is an ultraviolet divergent quantity, so one should renormalize it, using a special dependence of the bare quantities, such as the bare coupling constant $G\equiv G(\Lambda)$, on the cutoff parameter Λ (Λ restricts the integration region in the divergent integrals, $|p_1|<\Lambda$). The detailed discussion of this procedure was performed, for example, in papers [31, 32]. The main stages of it are the following. First, we transform the expression for the TDP (15),

$$\Omega_{\mu\mu_I}(M,\Delta)=V_0(M,\Delta)-\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\{E_\Delta^++E_\Delta^--2\sqrt{p_1^2+M^2+\Delta^2}+(\mu-E_\Delta^+)\theta(\mu-E_\Delta^+)+(\mu-E_\Delta^-)\theta(\mu-E_\Delta^-)\}. \tag{19}$$

The integral in the above expression is convergent and $V_0(M,\Delta)$ looks like

$$V_0(M,\Delta)\equiv\Omega_{\mu\mu_I}(M,\Delta)\Big|_{\mu=\mu_I=0}=\frac{M^2+\Delta^2}{4G}-2\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\sqrt{p_1^2+M^2+\Delta^2}. \tag{20}$$

Second, it is evident that the effective potential (20) [3] is a divergent quantity. So, to renormalize it we cut the integration region in (20), $|p_1| < \Lambda$, and require the bare coupling constant $G \equiv G(\Lambda)$ to have the following form

$$\frac{1}{2G(\Lambda)} = \frac{2}{\pi}\int_0^\Lambda dp_1 \frac{1}{\sqrt{M_0^2+p_1^2}} = \frac{2}{\pi}\ln\left(\frac{\Lambda+\sqrt{M_0^2+\Lambda^2}}{M_0}\right). \tag{21}$$

Then in the limit $\Lambda \to \infty$ one can obtain a finite, i.e. renormalized, expression for $V_0(M,\Delta)$:

$$V_0(M,\Delta) = \frac{M^2+\Delta^2}{2\pi}\left[\ln\left(\frac{M^2+\Delta^2}{M_0^2}\right)-1\right]. \tag{22}$$

Since M_0 might be considered as a free model parameter, it follows from (21) and (22) that the renormalization procedure of the NJL$_2$ model is accompanied by the dimensional transmutation phenomenon. Indeed, in the initial unrenormalized expression (20) for $V_0(M,\Delta)$ the dimensionless coupling constant G is present, whereas after renormalization the effective potential (22) is characterized by a dimensional free model parameter M_0. Moreover, as it is clear from (22), the global minimum point of the affective potential $V_0(M,\Delta)$ lies just at the point $M = M_0$. So in vacuum, the chiral $SU_L(2)\times SU_R(2)$ symmetry of the NJL$_2$ model (1) is always spontaneously broken and the quantity M_0 might be treated as a dynamical quark mass (in vacuum).

Now, taking into account the expression (22) for $V_0(M,\Delta)$, we see that the TDP (19) is a finite renormalization invariant quantity which describes the properties of dense and isotopically asymmetric quark matter in an infinite volume.

For the further analysis we need also the expressions for the quark number density n_q and isospin density n_I in the phase with gaps M and Δ which follow directly from (19),

$$\begin{aligned} n_q &\equiv -\frac{\partial\Omega_{\mu\mu_I}}{\partial\mu} = \int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\Big\{\theta(\mu-E_\Delta^+)+\theta(\mu-E_\Delta^-)\Big\},\\ n_I &\equiv -\frac{\partial\Omega_{\mu\mu_I}}{2\partial\nu} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\Big\{\frac{E+\nu}{E_\Delta^+}\theta(E_\Delta^+-\mu)-\frac{E-\nu}{E_\Delta^-}\theta(E_\Delta^--\mu)\Big\}, \end{aligned} \tag{23}$$

C. Compactification of the spatial coordinate

Since physical effects generally take place in a restricted space region, in the present paper we are going to study the influence of a finite size of the system (at zero temperature) on the thermodynamical properties of dense and isotopically asymmetric quark matter. To simulate a real situation, we put our (1+1)-dimensional system with Lagrangian (1) into a restricted space region of the form $0 \le x \le L$ (here x is the space coordinate). Then a corresponding modification of the TDP (19) is needed.

It is well-known that such a constraint on any physical system is equivalent to its investigation in a spacetime with nontrivial topology in which space coordinates are compactified. In our case it means that we can consider the model (1) in spacetime with the topology $R^1 \times S^1$ and with quantum fields satisfying some boundary conditions of the form

$$q(t,x+L) = e^{i\pi\alpha}q(t,x), \tag{24}$$

where $0 \le \alpha \le 2, L$ is the length of the circumference S^1, and the variable x means the path along it. Below, we shall use only two values of the parameter $\alpha : \alpha = 0$ for a periodic boundary condition and $\alpha = 1$ for the antiperiodic one.

As a consequence, to obtain the TDP $\Omega_{L\mu\mu_I}(M,\Delta)$ of the initial system placed in the restricted domain $0 \le x \le L$ and at nonzero chemical potentials $\mu \ne 0,\ \mu_I \ne 0$, one must simply replace the integration in (19) and (20) by an infinite series, according to the rule:

$$\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}f(p_1) \to \frac{1}{L}\sum_{n=-\infty}^{\infty}f(p_{1n});\quad p_{1n} = \frac{\pi}{L}(2n+\alpha),\quad n = 0,\pm1,\pm2,\dots \tag{25}$$

As a result, we obtain for the corresponding TDP $\Omega_{L\mu\nu}(M,\Delta)$ in the spacetime of the form $R^1 \times S^1$ the following expression

$$\begin{aligned}\Omega_{L\mu\mu_I}(M,\Delta) = V_L(\rho) - \frac{1}{L}\sum_{n=-\infty}^{\infty}\Big\{&E_{L\Delta n}^+ + E_{L\Delta n}^- - 2\sqrt{\rho^2+\frac{\pi^2}{L^2}(2n+\alpha)^2}\\ &+(\mu-E_{L\Delta n}^+)\theta(\mu-E_{L\Delta n}^+)+(\mu-E_{L\Delta n}^-)\theta(\mu-E_{L\Delta n}^-)\Big\},\end{aligned} \tag{26}$$

[3] Recall, the thermodynamic potential in vacuum, i.e. at $\mu = \mu_I = 0$, is usually called effective potential.

where $\rho = \sqrt{M^2 + \Delta^2}$,

$$E^{\pm}_{L\Delta n} = \sqrt{\left(\sqrt{M^2 + \frac{\pi^2}{L^2}(2n+\alpha)^2} \pm \nu\right)^2 + \Delta^2} \tag{27}$$

and

$$V_L(\rho) - V_L(0) = -\frac{\rho^2}{\pi}\ln\left(\frac{M_0 L}{4\pi}\right) - \frac{\rho^2\gamma}{\pi} - \frac{2}{L^2}\sqrt{\rho^2 L^2 + \pi^2\alpha^2} + \frac{2\pi\alpha}{L^2}$$
$$- \frac{2}{L^2}\sum_{n=1}^{\infty}\left[\sqrt{\pi^2(2n+\alpha)^2 + L^2\rho^2} + \sqrt{\pi^2(2n-\alpha)^2 + L^2\rho^2} - 4n\pi - \frac{\rho^2 L^2}{2n\pi}\right]. \tag{28}$$

Here $\gamma = 0.577...$ is the Euler constant. The expression (28) is the generalization of the vacuum effective potential (22) in the spacetime with trivial $R^1 \times R^1$-topology to the case of spacetime with nontrivial topology of the form $R^1 \times S^1$ (the detailed derivation of (28) is presented in Appendix of [31]). For further discussion, we need also the expression for the quark number density n_{qL} in the $R^1 \times S^1$ spacetime, which can be easily obtained from the TDP (26),

$$n_{qL} \equiv -\frac{\partial \Omega_{L\mu\mu_I}}{\partial \mu} = \frac{1}{L}\sum_{n=-\infty}^{\infty}\Big\{\theta(\mu - E^{+}_{L\Delta n}) + \theta(\mu - E^{-}_{L\Delta n})\Big\}. \tag{29}$$

Moreover, in what follows it will be convenient to use the dimensionless quantities:

$$\lambda = \frac{\pi}{LM_0}, \quad \tilde{\mu} = \frac{\mu}{M_0}, \quad \tilde{\nu} = \frac{\nu}{M_0} \equiv \frac{\mu_I}{2M_0}, \quad m = \frac{M}{M_0}, \quad \delta = \frac{\Delta}{M_0}. \tag{30}$$

III. PHASE STRUCTURE OF THE MODEL

In the present section we shall study the phase structure of the NJL$_2$ model (1) depending on three external parameters, $\tilde{\mu}, \tilde{\nu}$ and λ (30) in the case $T = 0$. In general, this first means that for each fixed values of $\tilde{\mu}, \tilde{\nu}$ and λ we should study the global minimum point of the TDP $\Omega_{\lambda\mu\mu_I}(M, \Delta)$ as a function of M, Δ (or dimensionless variables m, δ (30)) and then indicate the symmetry group of this point. Secondly, we must divide the three-dimensional space ($\tilde{\mu} \geq 0, \tilde{\nu} \geq 0, \lambda \geq 0$) into regions (phases), where each of them is composed by the points corresponding to the same symmetry group of the TDP global minimum point. Since this is a rather difficult problem, one can use an equivalent way. Instead of studying the above mentioned three-dimensional phase diagram, we will slice it into the planes $\lambda = const$ and then perform the above procedure only in the planes labeled by some values of λ. (Of course, the more values of λ are taken into consideration, the more exact is the obtained picture about the phase structure of the model.)

A. Particular case: $R^1 \times R^1$ spacetime ($\lambda = 0$)

Let us begin our study of the $(\tilde{\nu}, \tilde{\mu}, \lambda)$-phase diagram of the model (1) starting from the plane $\lambda = 0$ ($L = \infty$) that corresponds to the spacetime of trivial topology $R^1 \times R^1$. In this case the numerical investigations of the global minimum point (GMP) properties of the TDP (19) bring us to the following phase structure of the model shown in Fig. 1. There, in the corresponding $(\tilde{\nu}, \tilde{\mu})$-plane, one can see four phase regions denoted by PC as well as by the roman figures I, II and III, respectively.

In the region denoted by PC, i.e. in the charged pion condensation phase, each point $(\tilde{\nu}, \tilde{\mu})$ corresponds to the GMP of the form $(m = 0, \delta = 1)$. Hence, everywhere in this phase the relation $\tilde{\mu} < \tilde{\mu}_c < \delta$ is valid, since $\tilde{\mu}_c = 1/\sqrt{2}$. As a result, we see from (18) that all quasiparticles are gapped in the PC phase, i.e. a finite amount of energy is needed to create both u- and d-quarks. Moreover, it is easily seen from (23) that the quark number density n_q is identically zero in this phase. In contrast, the isospin density n_I is nonzero and equal to ν/π at each of the PC phase points.

Then, let us observe the properties of phases denoted by I, II, III. At each point of these phases the gap $\delta = 0$, but the gap $m = 0$ appears only in the phases I and III. As a result, in regions I and III the chirally $U_{I_3}(1) \times U_{AI_3}(1)$-symmetric phase with massless quarks is arranged. Contrary, in the region II, where the order parameter m is nonzero, this symmetry is spontaneously broken down to the isospin U_{I_3} subgroup. In this region the order parameter m is a smooth function vs $\tilde{\mu}$ and $\tilde{\nu}$ and tends to zero when $\tilde{\mu}$ and $\tilde{\nu}$ tend to infinity ($m \to 0$ when $\tilde{\mu} \to \infty$ and $\tilde{\nu} \to \infty$). We call this phase the normal quark matter phase, since here quarks dynamically acquire a mass which is equal to the order parameter m (multiplied by M_0), and space parity is not broken. Note also that the upper and lower boundaries of the region II tend asymptotically to the line $\tilde{\mu} = \tilde{\nu}$ from the upper- and lower sides, respectively. In the regions I

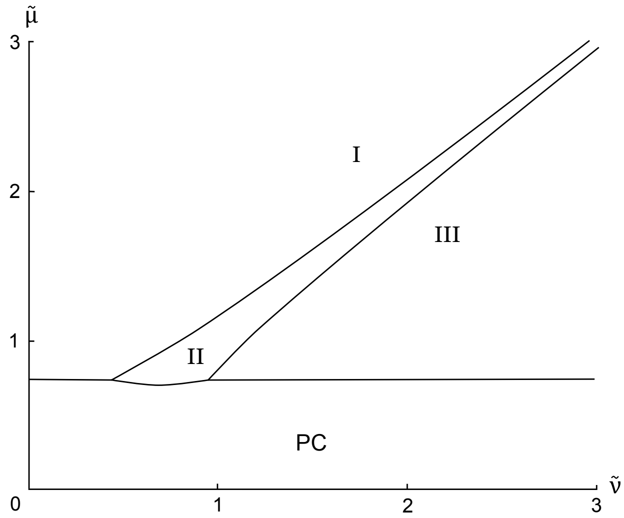

FIG. 1. The $(\tilde{\nu}, \tilde{\mu})$ phase portrait of the model at $\lambda = 0$ and $\tilde{\nu} > 0$. In the phases I and III the initial $U_{I_3}(1) \times U_{AI_3}(1)$ symmetry is not broken. In the phase II it is broken down up to $U_{I_3}(1)$, and in the pion condensation (PC) phase with *zero quark number density* it is spontaneously broken down to $U_{AI_3}(1)$.

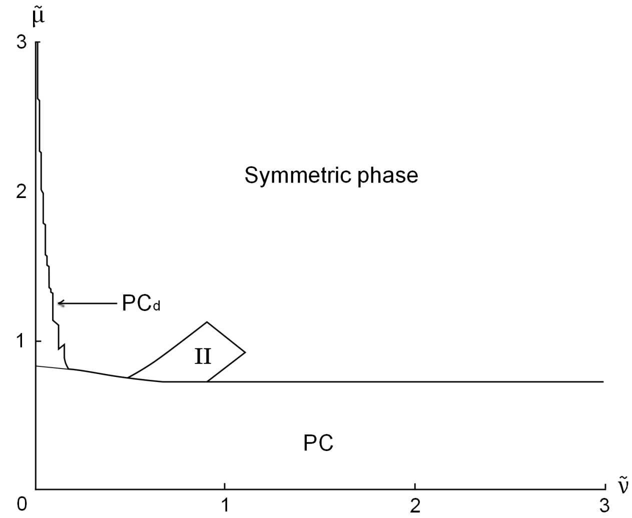

FIG. 2. *The periodic case*: The $(\tilde{\nu}, \tilde{\mu})$ phase portrait at $\lambda = 0.1$. PCd and PC mean the pion condensation phases with nonzero and zero quark number densities, respectively. II is the chirally non-symmetrical phase.

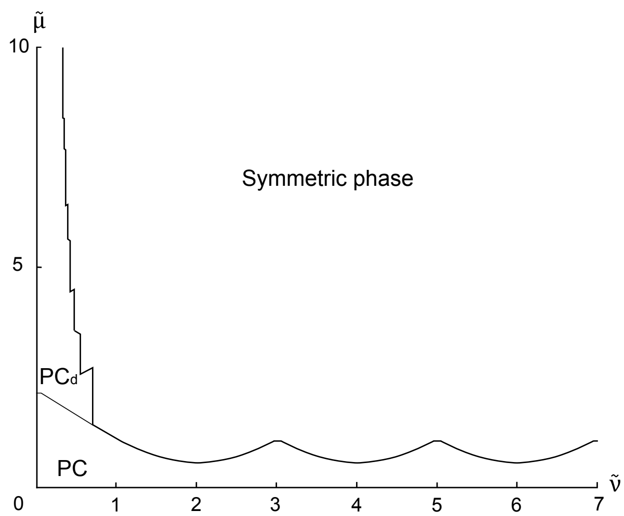

FIG. 3. *The periodic case*: The $(\tilde{\nu}, \tilde{\mu})$ phase portrait at $\lambda = 1$. PCd and PC mean the pion condensation phases with nonzero and zero quark number densities, respectively.

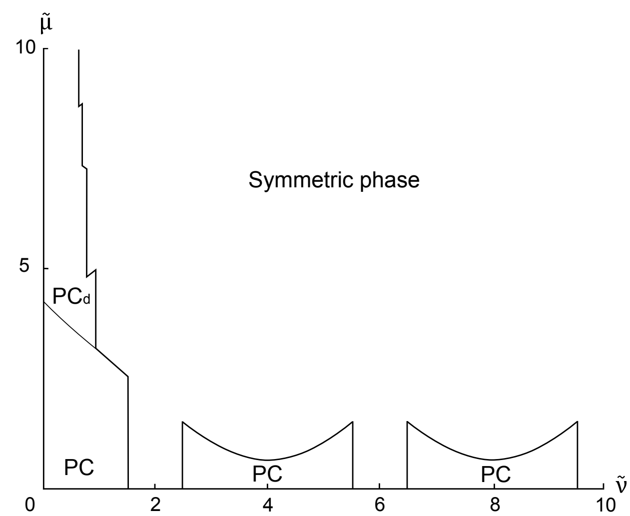

FIG. 4. *The periodic case*: The $(\tilde{\nu}, \tilde{\mu})$ phase portrait at $\lambda = 2$. PCd and PC mean the pion condensation phases with nonzero and zero quark number densities, respectively.

and III, as it follows from (23), the quark number density is $n_q = 2\mu/\pi$ and the isospin density is $n_I = \nu/\pi$. Finally, we would like to discuss the difference between I and III phases. It is evident that in phases I and III the dispersion relations (18) for quasiparticles have the following simple form:

$$p_{0u} = |p_1 - \nu| - \mu; \quad p_{0d} = p_1 + \nu - \mu. \tag{31}$$

Therefore, it is easy to check that in phase I both u- and d- quasiparticles are gapless. This means that to create these quarks costs no energy, i.e. at each fixed values of μ, ν there exist space momenta p_1^* and p_1^{**} such that $p_{0u}(p_1^*) = 0$ and $p_{0d}(p_1^{**}) = 0$. In contrast, in phase III only u-quasiparticles are gapless, but d-quarks are gapped.

B. General case: $R^1 \times S^1$ spacetime (periodic boundary conditions)

In the present section we consider both the phase structure and relevant properties of the model (1) at some particular nonzero values of λ in the case of periodic boundary conditions, i.e. at $\alpha = 0$ in (24). In terms of the dimensionless quantities (30) the TDP (26) has in the periodic case the following form:

$$\frac{\pi}{M_0^2}\Omega_{L\mu\mu_I}(M,\Delta) = (m^2+\delta^2)[\ln(4\lambda)-\gamma] - \lambda\sqrt{(m+\tilde{\nu})^2+\delta^2} - \lambda\sqrt{(m-\tilde{\nu})^2+\delta^2}$$
$$-2\lambda\sum_{n=1}^{\infty}\Big\{E^+_{n\alpha=0}+E^-_{n\alpha=0}-4n\lambda-\frac{m^2+\delta^2}{2n\lambda}+(\tilde{\mu}-E^+_{n\alpha=0})\theta(\tilde{\mu}-E^+_{n\alpha=0})+(\tilde{\mu}-E^-_{n\alpha=0})\theta(\tilde{\mu}-E^-_{n\alpha=0})\Big\}, \quad (32)$$

where $\gamma = 0.577...$ is again the Euler constant and

$$E^{\pm}_{n\alpha=0} = \sqrt{(\sqrt{m^2+(2n\lambda)^2}\pm\tilde{\nu})^2+\delta^2}. \quad (33)$$

We also need an expression for the quark number density n_{qL} in the periodic case which follows directly from (29) at $\alpha = 0$,

$$\tilde{n}_{qL} \equiv \frac{\pi}{M_0}n_{qL} = \lambda\theta(\tilde{\mu}-\sqrt{(m+\tilde{\nu})^2+\delta^2}) + \lambda\theta(\tilde{\mu}-\sqrt{(m-\tilde{\nu})^2+\delta^2})$$
$$+2\lambda\sum_{n=1}^{\infty}\Big\{\theta(\tilde{\mu}-E^+_{n\alpha=0})+\theta(\tilde{\mu}-E^-_{n\alpha=0})\Big\}, \quad (34)$$

where the zero mode terms are selected in an explicit form. The results of numerical investigations of the global minimum point (GMP) of the TDP (32) are presented in Figs. 2-4, where the phase portraits of the model are depicted at $\lambda = 0.1,\ 1,\ 2$, respectively. There, in the symmetric phase the global minimum of the TDP lies at the $U_{I_3}(1)\times U_{AI_3}(1)$ symmetric point $(m = 0, \delta = 0)$. The phase II corresponds to the $U_{I_3}(1)$ symmetric GMP of the form $(m \neq 0, \delta = 0)$. In this phase quarks are massive, and the quark number density n_{qL} is nonzero. In contrast to the case $\lambda = 0$ $(L = \infty)$, we see that at $\lambda \neq 0$ the phase II occupies a compact region in the phase diagram and completely vanishes at $\lambda > \lambda_p \approx 0.16$.

Moreover, one can see in Figs 2-4 two pion condensation phases, PC and PCd. They correpond to the GMP of the TDP (32) with $(m = 0, \delta \neq 0)$ and the $U_{AI_3}(1)$ symmetry group. Clearly, the gap δ of the pion condensation phases, as well as the gap m in the above mentioned phase II, depend on the external parameters, i.e. $\delta \equiv \delta(\tilde{\mu},\tilde{\nu},\lambda)$ (see Figs 5, 6). The main difference between these pion condensation phases is the following. In the PC phase the quark number density n_{qL} is equal to zero, whereas in the PCd phase this quantity is nonzero. The boundary between PC and PCd phases is defined by the zero mode term in (34) at $m = 0$ and $\delta = \delta(\tilde{\mu},\tilde{\nu},\lambda)$, i.e. it is the plot of the function $\tilde{\mu} = \tilde{\mu}(\tilde{\nu},\lambda)$ which is defined implicitly by the equation

$$\tilde{\mu} = \sqrt{\tilde{\nu}^2+\delta^2(\tilde{\mu},\tilde{\nu},\lambda)}. \quad (35)$$

Note, we have found the PCd phase at each arbitrary small $\lambda > 0$, whereas at $\lambda = 0$, i.e. in the ordinary (1+1)-dimensional Minkowsky spacetime, it is absent. *This means that a nontrivial spacetime topology promotes the creation of the charged pion condensed phase with a nonzero quark number density.* The behavior of both the gap $\delta(\tilde{\mu},\tilde{\nu},\lambda)$ and quark number density $\tilde{n}_{qL}$ (34) vs $\tilde{\mu}$ in the case of a periodic boundary condition is illustrated in Fig. 5 at $\lambda = 2$ and $\tilde{\nu} = 0.1$.

C. General case: $R^1 \times S^1$ spacetime (antiperiodic boundary conditions)

In the antiperiodic case, i.e. at $\alpha = 1$, the TDP (26) has the following form in terms of variables (30):

$$\frac{\pi}{M_0^2}\Omega_{L\mu\mu_I}(M,\Delta) = (m^2+\delta^2)[\ln(\lambda)-\gamma] - 2\lambda\sum_{n=0}^{\infty}\Big\{E^+_{\alpha=1}+E^-_{\alpha=1}-2(2n+1)\lambda-\frac{m^2+\delta^2}{(2n+1)\lambda}$$
$$+(\tilde{\mu}-E^+_{\alpha=1})\theta(\tilde{\mu}-E^+_{\alpha=1})+(\tilde{\mu}-E^-_{\alpha=1})\theta(\tilde{\mu}-E^-_{\alpha=1})\Big\}, \quad (36)$$

where

$$E^{\pm}_{\alpha=1} = \sqrt{(\sqrt{m^2+(2n+1)^2\lambda^2}\pm\tilde{\nu})^2+\delta^2}. \quad (37)$$

The investigation of the TDP (36) leads to the following conclusions about the phase structure of the model (1) (see Figs 7–9): First, as in the case with periodic boundary conditions for quark fields, at $\lambda \neq \infty$ the chirally non-symmetrical phase II in the antiperiodic case occupies a compact region in the $(\tilde{\nu},\tilde{\mu})$–plane. In addition, this phase

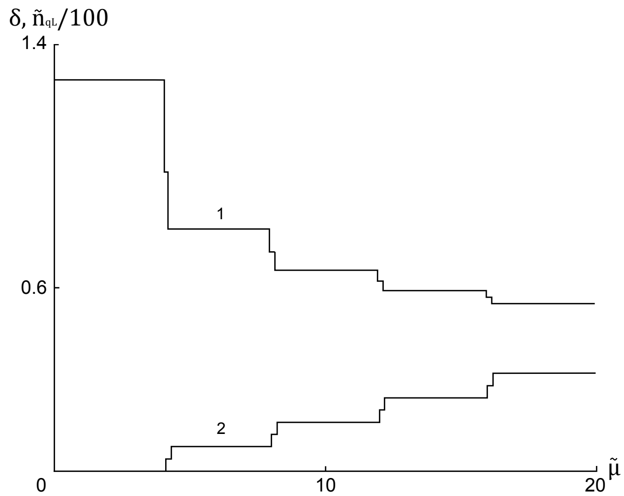

FIG. 5. *The periodic case*: The behavior of the gap δ (curve 1) and quark number density $\tilde{n}_{qL}$ (34) (curve 2) vs $\tilde{\mu}$ at $\lambda = 2$, $\tilde{\nu} = 0.1$.

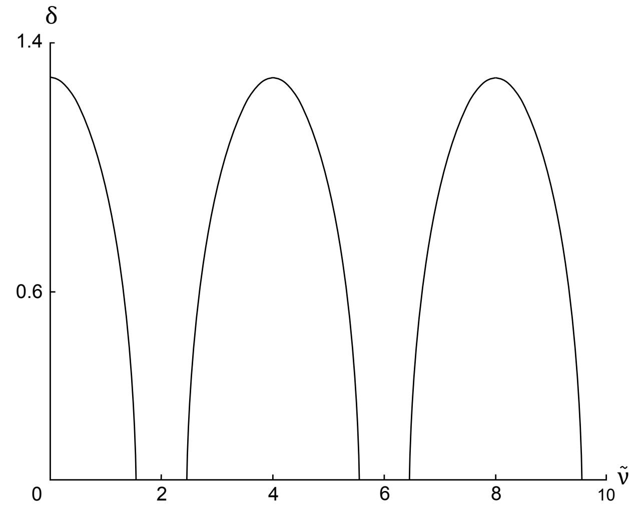

FIG. 6. *The periodic case*: The behavior of the gap δ vs $\tilde{\nu}$ at $\lambda = 2$ and arbitrary $\tilde{\mu} \in (0, 0.7)$.

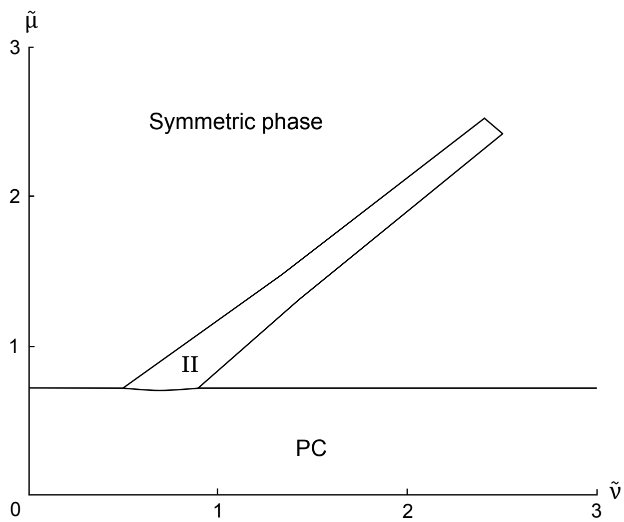

FIG. 7. *The antiperiodic case*: The phase portrait at $\lambda = 0.1$. PC again means the pion condensation phase with zero quark number densities. II is the chirally non-symmetric phase.

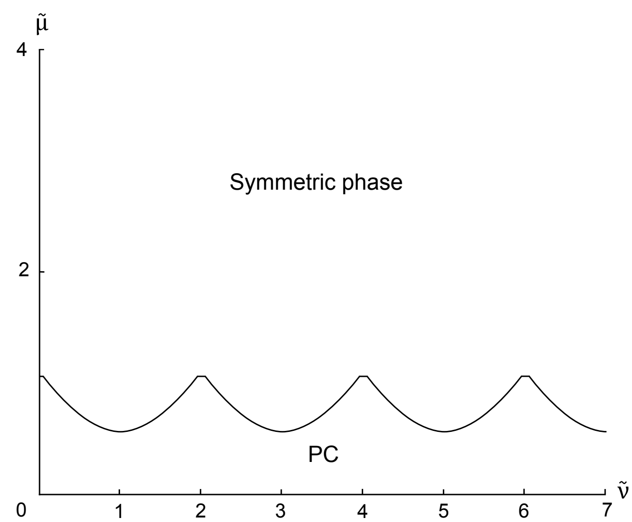

FIG. 8. *The antiperiodic case*: The phase portrait at $\lambda = 1$. PC means the pion condensation phase with zero quark number densities.

is realized only at $\lambda < \lambda_a \approx 0.66$. Secondly, in the antiperiodic case the phase portrait of the model contains only the pion condensed phase with zero quark number density at arbitrary λ–values, i.e. a PC phase with nonzero quark number densities is absent. The behavior of the gap δ vs $\tilde{\nu}$ at $\lambda = 2$ and arbitrary fixed value of $\tilde{\mu} \in (0, 1)$ is depicted in Fig. 10.

Finally, note that the above results, obtained both in the periodic and antiperiodic cases, refer strictly to the case $\nu > 0$. A detailed study of GN-type models in spacetime with nontrivial topology in the case of $\mu \neq 0$, $\nu = 0$ was made in [33].

IV. SUMMARY AND DISCUSSION

In this paper we have studied the so-called charged pion condensation (PC) phenomenon which might be observed in dense baryonic matter. Since we were going to pay special attention to the role of finite volume and to concentrate our attention on the principle properties of this effect, we have restricted ourselves to the (1+1)-dimensional NJL_2 model (1) with finite values of quark number μ and isospin μ_I chemical potentials at zero temperature. Moreover, our consideration was performed in the leading order of the large N_c-expansion.

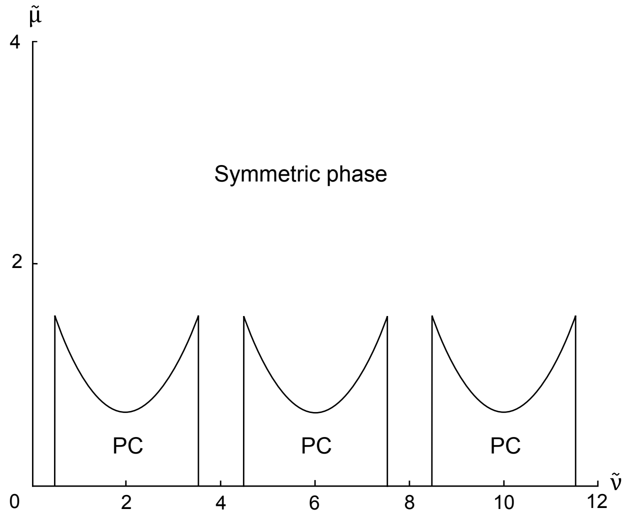

FIG. 9. *The antiperiodic case*: The phase portrait at $\lambda = 2$. PC means the pion condensation phase with zero quark number densities.

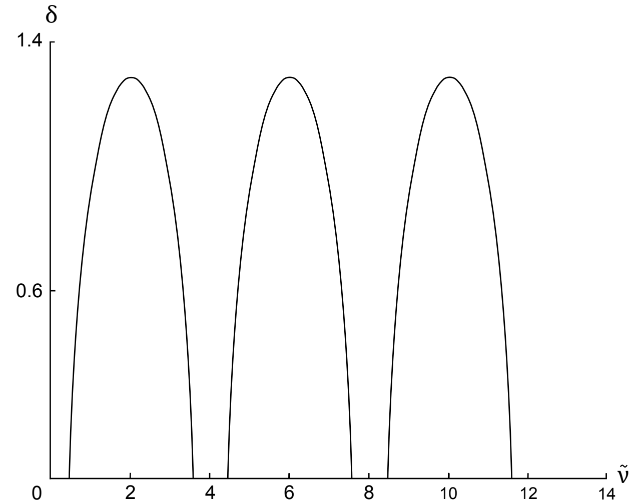

FIG. 10. *The antiperiodic case*: The behavior of the gap δ vs $\tilde{\nu}$ at $\lambda = 2$ and arbitrary $\tilde{\mu} \in (0, 0.7)$.

Recall that the charged PC phenomenon was studied recently in the framework of some QCD-like effective theories such as NJL-type models or chiral effective theories in usual (3+1)-dimensional Minkowski spacetime [10–13]. However, the existence of the PC phase with *nonzero baryon (quark number) density*, which we have called the PCd phase, was predicted there without any confidence. Indeed, for some values of model parameters (the coupling constant G and cutoff parameter Λ, etc.) the PCd phase is allowed by the NJL-type models. However, for other physically interesting values of G and Λ the PCd phase is forbidden in the framework of these models [11]. Moreover, if the electric charge neutrality constraint is imposed, the pion condensation phenomenon depends strongly on the bare (current) quark mass values. In particular, it turns out that the PCd phase is forbidden in the framework of the NJL-type models if bare quark masses reach the physically acceptable values of $5 \div 10$ MeV (see [13]). In addition, recent investigations of the charged pion condensation phenomenon in terms of the (1+1)-dimensional massive/massless NJL_2 model in $R^1 \times R^1$ spacetime shows that the PCd phase is also absent there [32] (see also Sec. III A of the present paper).

In this paper we have placed the NJL_2 system (1) into a restricted spatial region or, equivalently, in the spacetime with nontrivial topology of the form $R^1 \times S^1$, where the space coordinate is compactified into a circumference of a finite length L. It turns out that at $\lambda = 0$ ($\lambda \sim 1/L$) the (μ_I, μ)-phase diagram of the model contains *the zero quark number density PC phase* as well as two symmetric phases, where the initial symmetry of the model remains intact, and the phase II with broken chiral symmetry (see Fig. 1). Then, if $\lambda > 0$, we have observed in the (μ_I, μ)-phase diagram a more physically interesting situation, since at periodic boundary conditions for quark fields and for rather small values of μ_I the PCd phase is realized there (see Figs 2-4). So we may conclude that *finite size promotes the appearence of the PC phase with nonzero baryonic density* at least in the framework of the NJL_2 model (1). This is one of the main results of our paper and we hope that such an effect also takes place in the more realistic case of (3+1)-dimensional spacetime. Taking into account the fact that Cooper pairing of quarks is also significantly facilitated at $L \neq \infty$ [20], it is reasonable to expect that finite size can affect key properties of any system in comparison with those obtained from the $L \to \infty$ limit (see, e.g., the papers [17, 18, 21], where this point of view is also supported).

Among other results of our paper, it is interesting to note that the chirally broken phase II exists on the (μ_I, μ)-phase diagrams only at $\lambda < \lambda_p \approx 0.16$ ($\lambda < \lambda_a \approx 0.66$) in the case of periodic (antiperiodic) boundary conditions. In contrast, the PC phase (PCd phase in the case of periodic boundary conditions) is realized at an arbitrary value of $\lambda > 0$.

Obviously, it would be also interesting to study the influence of finite-size effects on the existence of chiral density waves and on the PC phenomenon in some QCD-like models in (3+1)-dimensional spacetime.

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